Determinants of elliptic boundary problems for Dirac operators I. Local boundary conditions. *

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Abstract

We study functional determinants for Dirac operators on manifolds with boundary and discuss the ellipticity of boundary problems by using the Calderón projector. We give, for local boundary conditions, an explicit formula relating these determinants to the corresponding Green functions. We finally apply this result to the case of a bidimensional disk under bag-like conditions.

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1 Introduction

It is well known that functional determinants have wide application in Quantum and Statistical Physics. Typically, one faces the necessity of defining a regularized determinant for elliptic differential operators. In this context, the Dirac first order differential operator plays a central role.

Seeley's construction of complex powers of elliptic differential operators provides a powerful tool to regularize such determinants: the so called ζ-function method [12].

In the case of boundaryless manifolds, this construction has been largely studied and applied (see, for instance, [8] and references therein).

For manifolds with boundary, the study of complex powers was performed in [17, 18] for the case of local boundary conditions, while for the case of nonlocal conditions, this task is still in progress (see, for example, [11].)

In general, the regularized determinant turns out to be nonlocal and so, it cannot be expressed in terms of just a finite number of Seeley's coefficients. However, such determinant can always be obtained from the Green function in a finite number of steps involving these coefficients. For boundaryless manifolds this was proved in [9], while for a particular type of local boundary conditions the procedure was introduced in [6].

The aim of this paper is to give a rigorous proof of the validity of this assertion in the case of the Dirac operator under general local elliptic boundary conditions. In so doing, the explicit relationship between determinants and the corresponding Green functions will be derived.

Dirac operators defined on manifolds with boundaries have been the subject of a vast literature (see, for instance, [15, 14] and references therein), mainly concerning anomalies and index theorems. In these papers, the emphasis was put on nonlocal boundary conditions of the type introduced in [1]. We leave for a forthcoming publication the treatment of such conditions.

Less attention was devoted to local boundary conditions in physical literature. In fact, some problems do not even admit such conditions owing to *topological obstructions* . We find enlightening to make a detailed discussion of this point since, to our knowledge, it has not been done in this context.

The outline of this paper is as follows:

In Section 2 we study elliptic boundary problems for Dirac operators, by means of the Calderón projector [4]. We discuss in detail the topological obstructions, and consider a class of local boundary conditions giving rise to

elliptic boundary systems.

In Section 3 we review, for the sake of completeness, Seeley's construction for complex powers of pseudo - differential operators on manifolds with boundary.

In Section 4 we present the main result of the paper: A formula relating the determinant of the Dirac operator with its Green function is established.

In Section 5 an explicit computation of the determinant of a Dirac operator in a bidimensional disk with bag-like boundary conditions is given.

2 The Calderón projector and Elliptic boundary problems

Throughout this paper we will be concerned with boundary value problems associated to first order elliptic operators

$$D : C^{\infty}(M, E) \to C^{\infty}(M, F), \tag{1}$$

where M is a compact connected sub manifold of $\mathbf{R}^{\nu}$ of codimension zero with smooth boundary ∂M, and E and F are k-dimensional complex vector bundles over M.

In a collar neighborhood of ∂M in M, we will take coordinates $\bar{x} = (x, t)$, with t the inward normal coordinate and x local coordinates for ∂M (that is, $t > 0$ for points in $M \setminus \partial M$ and $t = 0$ on ∂M), and conjugated variables $\bar{\xi} = (\xi, \tau)$.

As stated in the Introduction, we will mainly consider the Euclidean Dirac operator. Let us recall that the free Euclidean Dirac operator $\not\partial$ is defined as

$$i \not\partial = \sum_{\mu=0}^{\nu-1} i\gamma_{\mu} \frac{\partial}{\partial x_{\mu}}, \tag{2}$$

where the matrices γ_{μ} satisfy

$$\gamma_{\mu}\gamma_{\alpha} + \gamma_{\alpha}\gamma_{\mu} = 2\delta_{\mu\alpha}, \tag{3}$$

and that, given a gauge potential $A = \{A_{\mu},\ \mu = 0, ..., \nu - 1\}$ on M, the coupled Dirac operator is defined as

$$D(A) = i \not\partial + \not A \tag{4}$$

with $\not{A} = \sum_{\mu=0}^{\nu-1} \gamma_\mu A_\mu$.

In Section 5, explicit computations are carried out for $\nu = 2$. We take, for this case, the representation for the $\gamma' s$ given by the Pauli matrices $\{\sigma_j,\ j = 1, 2, 3\}$

$$
\begin{gathered}
\gamma_0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\gamma_5 = -i\gamma_0\gamma_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{gathered} \tag{5}
$$

In this representation, fermions with positive chirality are of the form $\binom{\bullet}{0}$, and those with negative chirality of the form $\binom{0}{\bullet}$. So, the free Dirac operator can be written as:

$$
i\ \not{\partial} = i\ (\gamma_0\ \partial_0 + \gamma_1\ \partial_1) = 2i\ \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z^*} & 0 \end{pmatrix}, \tag{6}
$$

where $z = x_0 + i\ x_1$.

One of the most suitable tools for studying boundary problems is the Calderón projector Q [4]. For the case we are interested in, D of order 1 as in (1), Q is a (not necessarily orthogonal) projection from $[L^2(\partial M, E_{/\partial M})]$ onto the subspace $\{(T\varphi\ /\varphi \in \ker(D)\}$, being $T : C^\infty(M, E) \to C^\infty(\partial M, E_{/\partial M})$ the trace map.

Given any fundamental solution $K(\bar{x}, \bar{y})$ of D, the projector Q can be constructed in the following way: for $f \in C^\infty(\partial M, E_{/\partial M})$, one gets $\varphi \in \ker(D)$ by means of a Green formula involving $K(\bar{x}, \bar{y})$, and takes the limit of φ for $\bar{x} \to \partial M$.

As shown in [4], Q is a zero-th order pseudo differential operator and its principal symbol $q(x; \xi)$, that depends only on the principal symbol of $D, \sigma_1(D) = a_1(x, t; \xi, \tau)$, turns out to be the $k \times k$ matrix

$$
q(x; \xi) = \frac{1}{2\pi i} \int_\Gamma \left(a_1^{-1}(x, 0; 0, 1)\ a_1(x, 0; \xi, 0) - z \right)^{-1} dz, \tag{7}
$$

where Γ is any simple closed contour oriented clockwise and enclosing all poles of the integrand in $Im(z) < 0$.

Although Q is not uniquely defined, since one can take any fundamental solution K of D to construct it, the image of Q and its principal symbol $q(x;\xi)$ are independent of the choice of K [4].

We find enlightening to compute the principal symbol of the Calderón projector for the Dirac operator as in (4) directly from the definition of Q, instead of using (7).

Let $K(\bar{x}, \bar{y})$ be a fundamental solution of the Dirac operator $D(A)$ in a neighborhood of the region M$\subset \mathbf{R}^\nu$, i.e.

$$D^\dagger(A) K^\dagger(\bar{x}, \bar{y}) = \delta(\bar{x} - \bar{y}). \tag{8}$$

We can write

$$K(\bar{x}, \bar{y}) = K_0(\bar{x}, \bar{y}) + R(\bar{x}, \bar{y}) \tag{9}$$

where $K_0(\bar{x}, \bar{y})$ is a fundamental solution of $i \not{\partial}$ and $|R(\bar{x}, \bar{y})|$ is O$(1/|\bar{x} - \bar{y}|^{\nu-2})$ for $\bar{x} - \bar{y} \sim 0$. Since $(i \not{\partial})^2 = -\triangle$, it is easy to obtain $K_0(\bar{x}, \bar{y})$ from the well known fundamental solution of the Laplacian, so

$$K_0(\bar{x}, \bar{y}) = -\, i \, \frac{\Gamma(\nu/2)}{2\, \pi^{\nu/2}} \; \frac{(\bar{\not{x}} - \bar{\not{y}})}{|\bar{x} - \bar{y}|^\nu}. \tag{10}$$

For f a smooth function on ∂M,

$$Q f(x) = -i \lim_{\bar{x} \to \partial M} \int_{\partial M} K(\bar{x}, y) \;\; \not{n} \; f(y) \; d\sigma_y, \tag{11}$$

where $\not{n} = \sum_l \gamma_l \; n_l$, and $n = (n_l)$ is the unitary outward normal vector on ∂M. Note that, if $f = T\varphi$, with $\varphi \in \ker(D)$, the Green formula yields $Qf = f$, as required.

From (9), (10) and (11) one gets

$$Q f(x) = \frac{1}{2} f(x) - i \; P.V. \int_{\partial M} K(x, y) \;\; \not{n} \; f(y) \; d\sigma_y \tag{12}$$

In order to see that the principal value in (12) makes sense, note that

$$P.V. \int_{\partial M} K_0(x, y) \;\; \not{n} \; f(y) \; d\sigma_y \tag{13}$$

is meaningful since

$$-\, i \int_{\partial M} K_0(\bar{x}, y) \;\; \not{n} \;\; d\sigma_y = Id_{k \times k}, \; \forall \bar{x} \in M. \tag{14}$$

In fact, any $\varphi(\bar{x}) \equiv constant$ is a solution of $i \not{\partial}\varphi = 0$ in M.

Then

$$\begin{aligned} Qf(x) = \tfrac{1}{2} f(x) - i \ P.V. \textstyle\int_{\partial M} K_0(x,y) \ \not{n} \ f(y) \ d\sigma_y \\ -i \ \textstyle\int_{\partial M} R(x,y) \ \ \not{n} \ f(y) \ d\sigma_y \end{aligned} \tag{15}$$

For the calculus of the principal symbol, we write the second term in the r.h.s. of (15) in local coordinates on ∂M ,

$$- iP.V. \int_{\mathbf{R}^{\nu-1}} \frac{\Gamma(\nu/2)}{2 \ \pi^{\nu/2}} \ \frac{(x-y)_j}{|x-y|^\nu} \ \gamma_j \ \gamma_n f(y) \ dy = \frac{1}{2} \ \gamma_j \ \gamma_n \ \mathbf{R}_j(f)(x), \tag{16}$$

where $\mathbf{R}_j(f)$ is the j-th Riesz transform of f. The symbol of the operator in (16) is (see for example [19])

$$\frac{1}{2} i \ \gamma_j \ \gamma_n \frac{\xi_j}{|\xi|} \ = \frac{1}{2} i \ \frac{\not{\xi}}{|\xi|} \ \ \not{n}. \tag{17}$$

The last term in the r.h. side of (15) is a pseudo - differential operator of order ≤ -1, because of the local behavior of $R(x,y)$, and then it does not contribute to the calculus of the principal symbol we are carrying out. Then, coming back to global coordinates, we finally obtain

$$q(x;\xi) = \frac{1}{2}(Id_{k\times k} + i \ \frac{\not{\xi}}{|\xi|} \ \ \not{n}). \tag{18}$$

In order to get the *rank* of this matrix, note that

$$\begin{aligned} q(x;\xi) \ q(x;\xi) = q(x;\xi) \\ tr \ q(x;\xi) = k/2, \end{aligned} \tag{19}$$

and consequently $rank \ q(x;\xi) = k/2$.

In particular for $\nu = 2$ and the $\gamma' s$ as in (5), we obtain

$$q(x;\xi) = \begin{pmatrix} H(\xi) & 0 \\ 0 & H(-\xi) \end{pmatrix} \tag{20}$$

$\forall x \in \partial M$, with $H(\xi)$ the Heaviside function.

According to Calderón [4], elliptic boundary conditions can be defined in terms of $q(x;\xi)$, the principal symbol of the projector Q.

Definition 1:

Let us assume that the *rank* of $q(x;\xi)$ is a constant r (as is always the case for $\nu \geq 3$ [4]).

A zero order pseudo differential operator $B : [L^2(\partial M, E_{/\partial M})] \to [L^2(\partial M, G)]$, with G an r dimensional complex vector bundle over ∂M, gives rise to an *elliptic boundary condition* for a first order operator D if, $\forall \xi : |\xi| \geq 1$,

$$rank(b(x;\xi)\ q(x;\xi)) = rank(q(x;\xi)) = r \ , \tag{21}$$

where $b(x;\xi)$ coincides with the principal symbol of B for $|\xi| \geq 1$.

In this case we say that

$$\begin{cases} D\varphi = \chi \ \text{ in M} \\ \\ BT\varphi = f \text{ on } \partial\text{M} \end{cases} \tag{22}$$

is an *elliptic boundary problem*, and denote by D_B the closure of D acting on the sections $\varphi \in C^\infty(M, E)$ satisfying $B(T\varphi) = 0$.

In particular, condition (21) implies that, for each $s \in \mathbf{R}$, the image of $B \circ Q$ is a closed subspace of the Sobolev space $H^s(\partial M, G)$ having finite codimension [4].

An elliptic boundary problem as (22) has a solution $\varphi \in H^1(M, E)$ for any (χ, f) in a subspace of $L^2(M, E) \times H^{1/2}(\partial M, G)$ of finite codimension. Moreover, this solution is unique up to a finite dimensional kernel [4]. In other words, the operator

$$(D, BT) : H^1(M, E) \to L^2(M, E) \times H^{1/2}(\partial M, G) \tag{23}$$

is Fredholm.

For $\nu = 2$, Definition 1 does not always apply. For instance, for the two dimensional chiral Euclidean Dirac operator

$$D = 2i\frac{\partial}{\partial z^*} \ , \tag{24}$$

acting on sections with positive chirality and taking values in the subspace of sections with negative one, it is easy to see from (20) that

$$q(x;\xi) = H(\xi). \tag{25}$$

Then, the *rank* of $q(x;\xi)$ is not constant. In fact,

$$rank\ q(x;\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ 1 & \text{if } \xi > 0 \end{cases}. \tag{26}$$

However, for the (full) two dimensional Euclidean Dirac operator

$$D(A) = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \tag{27}$$

we get from (19) that $rank\ q(x;\xi) = 2/2 = 1\ \forall \xi \neq 0$, and so Definition 1 does apply.

When B is a local operator, Definition 1 yields the classical local elliptic boundary conditions, also called Lopatinsky-Shapiro conditions (see for instance [13]) .

For Euclidean Dirac operators on $\mathbf{R}^\nu$, $E_{/\partial M} = \partial M \times \mathbf{C}^k$, and local boundary conditions arise when the action of B is given by the multiplication by a $\frac{k}{2} \times k$ matrix of functions defined on ∂M.

Owing to *topological obstructions,* chiral Dirac operators in even dimensions do not admit local elliptic boundary conditions (see for example [2]). For instance, in four dimensions, by choosing the γ-matrices at $x = (x_1, x_2, x_3) \in \partial M$ as

$$\gamma_n = i \begin{pmatrix} 0 & Id_{2\times 2} \\ -Id_{2\times 2} & 0 \end{pmatrix} \quad \text{and} \quad \gamma_{\rm j} = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } {\rm j} = 1,2,3, \tag{28}$$

the principal symbol of the Calderón projector (18) associated to the full Dirac operator turns out to be

$$q(x;\xi) = \frac{1}{2} \begin{pmatrix} Id_{2\times 2} + \dfrac{\xi.\sigma}{|\xi|} & 0 \\ 0 & Id_{2\times 2} - \dfrac{\xi.\sigma}{|\xi|} \end{pmatrix}. \tag{29}$$

Thus, from the left upper block, one gets for the chiral Dirac operator

$$q_{ch}(x;\xi) = \frac{1}{2} \begin{pmatrix} 1 + \dfrac{\xi_3}{|\xi|} & \dfrac{\xi_1 - i\xi_2}{|\xi|} \\ \dfrac{\xi_1 + i\xi_2}{|\xi|} & 1 - \dfrac{\xi_3}{|\xi|} \end{pmatrix}. \tag{30}$$

So $q_{ch}(x;\xi)$ is a hermitian idempotent 2×2 matrix with $rank = 1$. If one had a local boundary condition with principal symbol $b(x) = (\beta_1(x), \beta_2(x))$, according to Definition 1, it should be $rank(b(x)\ q_{ch}(x;\xi)) = 1,\ \forall\xi \neq 0$. However, it is easy to see that for

$$\xi_1 = \frac{-2\beta_1\beta_2}{\beta_1^2+\beta_2^2}, \qquad \xi_2 = 0 \quad \text{and} \quad \xi_3 = \frac{\beta_2^2-\beta_1^2}{\beta_1^2+\beta_2^2}, \tag{31}$$

$rank(b(x)\ q_{ch}(x;\xi)) = 0$. Equivalently, this means that $q_{ch}(x;\xi)$ is not deformable through idempotents of $rank = 1$ to a $\xi-$independent matrix function . This is an example of the above discussed topological obstructions.

Nevertheless, it is easy to see that local boundary conditions can be defined for the full, either free or coupled, Euclidean Dirac operator

$$D(A) = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$$

on M. For instance, we see from (20) and (21) that for $\nu = 2$, the operator B defined as

$$B\begin{pmatrix} f \\ g \end{pmatrix} = (\beta_1(x), \beta_2(x))\begin{pmatrix} f \\ g \end{pmatrix} \tag{32}$$

yields a local elliptic boundary condition for every couple of nowhere vanishing functions $\beta_1(x)$ and $\beta_2(x)$ on ∂M.

A type of non-local boundary conditions, to be consider in a forthcoming publication, is related to the ones defined and analyzed by M. Atiyah , V. Patodi and I. Singer in [1] for a wide class of first order Dirac-like operators, including the Euclidean chiral case. Near ∂M such operators can be written as

$$\sigma\ (\partial_t + A), \tag{33}$$

where σ is an isometric bundle isomorphism $E \to F$, and $A : L^2(\partial M, E_{/\partial M}) \to L^2(\partial M, E_{/\partial M})$ is self adjoint. The operator P_{APS} defining the boundary condition is the orthogonal projection onto the closed subspace of $L^2(\partial M, E_{/\partial M})$ spanned by the eigenfunctions of A associated to non negative eigenvalues.

The projector P_{APS} is a zero order pseudo differential operator and its principal symbol coincides with the one of the corresponding Calderón projector [3].

The problem (22) with $B = P_{APS}$ has a solution $\varphi \in H^1(M,E)$ for any (χ, f) with χ in a finite codimensional subspace of $L^2(M,E)$ and f in the

intersection of $H^{1/2}(\partial M, E_{/\partial M})$ with the image of P_{APS}. The solution is unique up to a finite dimensional kernel. Note that, since the codimension of $P_{APS}\ [L^2(\partial M, E_{/\partial M})]$ is not finite, the operator

$$(D, P_{APS}T) : H^1(M,E) \to L^2(M,E) \times H^{1/2}(\partial M, E_{/\partial M}) \tag{34}$$

is not Fredholm.

Definition 1 does not encompass Atiyah, Patodi and Singer (A.P.S.) conditions since P_{APS} takes values in $L^2(\partial M, E_{/\partial M})$ instead of $L^2(\partial M, G)$, with G a r dimensional vector bundle ($r = rank\ q(x;\xi)$), as required in that definition. However, it is possible to define elliptic boundary problems according to Definition 1 by using conditions *à la* APS. For instance, the following self-adjoint boundary problem for the two-dimensional full Euclidean Dirac operator is elliptic:

$$\begin{aligned} \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0 \ \text{ in M}, \\ (P_{APS}, \sigma(I - P_{APS})\ \sigma^*) \begin{pmatrix} f \\ g \end{pmatrix} = 0 \ \text{ on } \partial\text{M}. \end{aligned} \tag{35}$$

In fact, as mentioned above, the principal symbol of P_{APS} is equal to the principal symbol of the Calderón projector associated to D. So, from (25) we get $\sigma_0(P_{APS}) = H(\xi)$. By taking adjoints we obtain $\sigma_0(\sigma(I - P_{APS})\ \sigma^*) = H(-\xi)$. Then, the principal symbol of $B = (P_{APS}, \sigma(I - P_{APS})\ \sigma^*)$ is $b(x;\xi) = (H(\xi), H(-\xi))$ and satisfies

$$rank(b(x;\xi)\ q(x;\xi)) = rank(q(x;\xi)) \qquad \forall \xi \neq 0. \tag{36}$$

3 Complex powers and regularized determinants for elliptic boundary problems.

In this section we describe Seeley's construction of the complex powers of the operator D under local elliptic boundary condition B.

We will denote by $\sigma'(D)$ the partial symbol of D, i.e. the symbol $\sigma(D)$ evaluated at $t = 0$ and $\tau = -i\partial_t$.

Definition 2:

The elliptic boundary problem (22) admits a cone of Agmon's directions if there is a cone Λ in the λ-complex plane such that

1) $\forall \bar{x} \in M$, $\forall \bar{\xi} \neq 0$, Λ contains no eigenvalues of the matrix $\sigma_1(D)(\bar{x}, \bar{\xi})$.

2) $\forall \xi : |\xi| \geq 1$, $rank(b(x;\xi)\ q(\lambda)(x;\xi)) = rank(\ q(\lambda)(x;\xi))$, $\forall \lambda \in \Lambda$, where $q(\lambda)$ denotes the principal symbol of the Calderón projector $Q(\lambda)$ associated to $D - \lambda I$, with λ included in $\sigma_1(D - \lambda I)$ (i.e. considering λ of degree one in the expansion of $\sigma(D - \lambda I)$ in homogeneous functions) [10] [17].

Condition 2 is equivalent to the following:

$\forall \lambda \in \Lambda$, $\forall x \in \partial M$, $\forall g \in \mathbf{C}^r$, the initial value problem

$$\sigma_1'(D)(x;\xi)\ u(t) = \lambda\ u(t)$$

$$b(x;\xi)\ u(t)|_{t=0} = g$$

has, for each $\xi \neq 0$, a unique solution satisfying $\lim_{t\to\infty} u(t) = 0$. This is the form under which this condition is stated in [17].

An expression for $q(\lambda)(x;\xi)$ is obtained from (7):

$$q(\lambda)(x;\xi) = \frac{1}{2\pi i}\int_\Gamma (a_1^{-1}(x,0;0,1;0)\ a_1(x,0;\xi,0;\lambda) - z)^{-1} dz, \tag{37}$$

where $a_1(x,t;\xi,\tau;\lambda) = \sigma_1(D - \lambda I)$, with λ considered of degree one as stated above.

As an example, we now compute $q(\lambda)(x;\xi)$ for the two-dimensional Dirac operator in (27) on a disk. In polar coordinates the principal symbol of $D(A) - \lambda I$ is

$$a_1(\theta,t;\xi,\tau;\lambda) = \tau\ \gamma_r - \xi\ \gamma_\theta - \lambda\ Id_{2\times 2}, \tag{38}$$

where

$$\gamma_r = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \qquad \gamma_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}. \tag{39}$$

So

$$q(\lambda)(x;\xi) = \frac{1}{2\pi i}\int_\Gamma \Big(\gamma_r(-\xi\ \gamma_\theta - \lambda\ Id_{2\times 2}) - z\Big)^{-1} dz$$

$$= \frac{1}{2\sqrt{\xi^2-\lambda^2}} \begin{pmatrix} \xi + \sqrt{\xi^2-\lambda^2} & -i\lambda e^{-i\theta} \\ -i\lambda e^{i\theta} & -\xi + \sqrt{\xi^2-\lambda^2} \end{pmatrix}. \tag{40}$$

Note that, for $\lambda = 0$, (20) is recovered.

Henceforth, we assume the existence of an Agmon's cone Λ. Moreover, we will consider only boundary conditions B giving rise to a discrete spectrum $sp(D_B)$. Note that, this is always the case for elliptic boundary problems unless $sp(D_B)$ is the whole complex plane (see, for instance,[13]). Now, for $|\lambda|$ large enough, $sp(D_B)\cap\Lambda$ is empty, since there is no λ in $sp(\sigma_1(D_B))\cap\Lambda$. Then, $sp(D_B)\cap\Lambda$ is a finite set.

For $\lambda \in \Lambda$ not in $sp(D_B)$, an asymptotic expansion of the symbol of $R(\lambda) = (D_B - \lambda I)^{-1}$ can be explicitly given [17]:

$$\sigma(R(\lambda)) \sim \sum_{j=o}^{\infty} c_{-1-j} - \sum_{j=o}^{\infty} d_{-1-j} \tag{41}$$

where the *Seeley coefficients* I c_{-1-j} and d_{-1-j} satisfy

$$\sum_{j=o}^{\infty} a_{1-j} \quad\circ\quad \sum_{j=0}^{\infty} c_{-1-j} = I \tag{42}$$

with a_{1-j} homogeneous of degree $1-j$ in $(\bar{\xi},\lambda)$ defined by

$$\sigma(D-\lambda I) = \sum_{j=o}^{\infty} a_{1-j}, \tag{43}$$

$\circ$ denoting the usual composition of homogeneous symbols, and

$$\begin{cases} \sigma'(D-\lambda)\ \circ\ \sum\limits_{j=o}^{\infty} d_{-1-j} = 0 \\ \sigma'(B)\ \circ\ \sum\limits_{j=o}^{\infty} d_{-1-j} = \sigma(B)\ \circ\ \sum\limits_{j=0}^{\infty} c_{-1-j} \quad \text{at}\quad \mathrm{t}=0 \\ \lim\limits_{t\to\infty}\ d_{-1-j} = 0 \end{cases} \tag{44}$$

where the terms of $\sigma'(D-\lambda)$ are grouped according to their degree of homogeneity in $(\frac{1}{t}, \xi, \partial_t, \lambda)$.

Note that condition 2) implies the existence and unicity of the solution of (44).

Written in more detail, the first line in (44) becomes [17]

$$a^{(1)} d_{-1-j} + \sum_{\substack{l<j \\ k-|\alpha|-1-l=-j}} \frac{i^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial \xi^\alpha} a^{(k)} \frac{\partial^\alpha}{\partial x^\alpha} d_{-1-l} = 0, \tag{45}$$

with

$$a^{(j)}(x,t;\xi,\partial_t;\lambda) = \sum_{l-k=j} \frac{t^k}{k!} \frac{\partial^k}{\partial t^k} a_l(x,t;\xi,\partial_t;\lambda)|_{t=0}, \tag{46}$$

while the second one becomes

$$\begin{aligned} b_0 d_{-1-j} + & \sum_{\substack{l<j \\ k-|\alpha|-1-l=-j}} \frac{i^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial \xi^\alpha} b_{-k} \frac{\partial^\alpha}{\partial x^\alpha} d_{-1-l} \\ = & \sum_{\substack{l<j \\ k-|\beta|-1-l=-j}} \frac{i^\beta}{\beta!} \frac{\partial^\beta}{\partial \bar{\xi}^\beta} b_{-k} \frac{\partial^\beta}{\partial \bar{x}^\beta} c_{-1-l}|_{t=0} \end{aligned} \tag{47}$$

It is worth noticing that, although

$$\sigma(R(\lambda)) = \sum_{j=0}^{\infty} c_{-1-j} \ , \tag{48}$$

is an asymptotic expansion of $\sigma(R(\lambda))$, the fundamental solution of $(D_B-\lambda)$ obtained by Fourier transforming (48) does not in general satisfy the required boundary conditions. The coefficients d_{-1-j} must be added to the expansion in order to correct this deficiency.

The coefficients c_{-1-j} $(x,t;\xi,\tau;\lambda)$ and $d_{-1-j}(x,t;\xi,\tau;\lambda)$ are meromorphic functions of λ with poles at those points where $\det[\sigma_1(D-\lambda)(x,t;\xi,\tau)]$ vanishes. The c_{-1-j}'s are homogeneous of degree $-1-j$ in (ξ,τ,λ) ; the d_{-1-j} 's are also homogeneous of degree $-1-j$, but in ($\frac{1}{t},\xi,\tau,\lambda$)[17].

Moreover, it can be proved from (41) that, for $\lambda \in \Lambda$,

$$\parallel R(\lambda) \parallel_{L^2} \leq C|\lambda|^{-1} \tag{49}$$

with C a constant [17, 10].

Estimate (49) allows for expressing the complex powers of D_B as

$$(D_B)^z = \frac{i}{2\pi} \int_\Gamma \lambda^z \; R(\lambda) \; d\lambda \tag{50}$$

for $Re\ z < 0$, where Γ is a closed path lying in Λ, enclosing the spectrum of (D_B) [18]. Note that such a curve Γ always exists for $sp(D_B) \cap \Lambda$ finite.

For $Re\ z \geq 0$, one defines

$$(D_B)^z = (D)^l \circ (D_B)^{z-l} \; , \tag{51}$$

for l a positive integer such that $Re\ (z-l) < 0$.

If $Re(z) < -\nu$, the power $(D_B)^z$ is an integral operator with continuous kernel $J_z(x,t;y,s)$ and, consequently, it is trace class (for an operator of order ω, this is true if $Re(z) < -\frac{\nu}{\omega}$). As a function of z, $Tr(D_B)^z$ can be meromorphicaly extended to the whole complex plane $\mathbf{C}$, with only single poles at $z = j - \nu,\ j = 0,1,2,...$ and vanishing residues when $z = 0,1,2,...$ (for an operator of order ω , there are only single poles at $z = \frac{j-\nu}{\omega},\ j = 0,1,2,...$, with vanishing residues at $z = 0,1,2,...$) [18]. Throughout this paper, analytic functions and their meromorphic extensions will be given the same name.

The function $Tr(D_B)^z$ is usually called $\zeta_{(D_B)}(-z)$ because of its similarity with the classical Riemann ζ-function: if $\{\lambda_j\}$ are the eigenvalues of D_B, $\{\lambda_j^z\}$ are the eigenvalues of $(D_B)^z$; so $Tr(D_B)^z = \sum \lambda_j^z$ when $(D_B)^z$ is a trace class operator.

A regularized determinant of D_B can then be defined as

$$Det\ (D_B) = \exp[-\frac{d}{dz}\ Tr\ (D_B)^z]|_{z=0}. \tag{52}$$

Now, let $D(\alpha)$ be a family of elliptic differential operators on M sharing their principal symbol and analytically depending on α. Let B give rise to an elliptic boundary condition for all of them, in such a way that $D(\alpha)_B$ is invertible and the boundary problems they define have a common Agmon's cone. Then, the variation of $Det\ D(\alpha)_B$ with respect to α is given by (see, for example, [1, 7])

$$\frac{d}{d\alpha} \ln\ Det\ D(\alpha)_B = \frac{d}{dz} \left[\ z\ Tr\{\frac{d}{d\alpha}\,(D(\alpha)_B)\ \ (D(\alpha)_B)^{z-1}\} \right]_{z=0}. \tag{53}$$

Note that, under the assumptions made, $\frac{d}{d\alpha}(D(\alpha)_B)$ is a multiplicative operator.

Although $J_z(x,t;x,t;\alpha)$, the kernel of $(D(\alpha)_B)^z$ evaluated at the diagonal, can be extended to the whole z-complex plane as a meromorphic function, the r.h.s. in (53) cannot be simply written as the integral over M of the finite part of

$$tr\{\frac{d}{d\alpha}(D(\alpha)_B)\ \ J_{z-1}(x,t;x,t;\alpha)\} \tag{54}$$

at $z=0$ (where tr means matrix trace). In fact, $J_{z-1}(x,t;x,t;\alpha)$ is in general non integrable in the variable t near ∂M for $z \approx 0$.

Nevertheless, an integral expression for $\frac{d}{d\alpha}\ln\ \ Det\ D(\alpha)_B$ will be constructed in Section 4, from the integral expression for $Tr(D(\alpha)_B)^{z-1}$ holding in a neighborhood of $z=0$ and obtained in the following way [18]:

if $T>0$ is small enough, the function $j_z(x;\alpha)$ defined as

$$j_z(x;\alpha)\ = \int_0^T J_z(x,t;x,t;\alpha)\ dt \tag{55}$$

for $Re\ z < 1-\nu$, admits a meromorphic extension to $\mathbf{C}$ as a function of z. So, if V is a neighborhood of ∂M defined by $t<\epsilon$, with ϵ small enough, $Tr(D(\alpha)_B)^{z-1}$ can be written as the finite part of

$$\int_{M/V} tr\ J_{z-1}(x,t;x,t;\alpha)\ dxdt + \int_{\partial M} tr\ j_{z-1}(x;\alpha)\ dx\ , \tag{56}$$

where a suitable partition of the unity is understood.

4 Green functions and determinants

In this section, we will give an expression for $\frac{d}{d\alpha}\ln\ \ Det[D(\alpha)_B]$ in terms of $G_B(x,t;y,s;\alpha)$, the Green function of $D(\alpha)_B$ (i.e., the kernel of the operator $[D(\alpha)_B]^{-1}$).

With the notation of the previous Section, (53) can be rewritten as:

$$\frac{d}{d\alpha}\ln\ \ Det\ D(\alpha)_B = \underset{z=0}{F.P.}\ \int_M tr\left[\frac{d}{d\alpha}(D(\alpha)_B)\ \ J_{-z-1}(x,t;x,t;\alpha)\right]\ d\bar{x}\ , \tag{57}$$

where the r.h.s. must be understood as the finite part of the meromorphic extension of the integral at $z = 0$.

The finite part of $J_{-z-1}(x,t;x,t;\alpha)$ at $z = 0$ does not coincide with the regular part of $G_B(x,t;y,s;\alpha)$ at the diagonal, since the former is defined through an analytic extension.

However, we will show that there exists a relation between them, involving a finite number of Seeley's coefficients. In fact, for boundaryless manifolds this problem has been studied in [9], by comparing the iterated limits $F.P. \lim\limits_{z\to -1}\{\lim\limits_{\bar{y}\to\bar{x}} J_z(x,t;y,s;\alpha)\}$ and $R.P. \lim\limits_{\bar{y}\to\bar{x}}\{\lim\limits_{z\to -1} J_z(x,t;y,s;\alpha)\} = R.P. \lim\limits_{\bar{y}\to\bar{x}} G_B(x,t;y,s;\alpha)$.

In the case of manifolds with boundary, the situation is more involved owing to the fact that the finite part of the extension of $J_z(x,t;x,t;\alpha)$ at $z = -1$ is not integrable near ∂M . (A first approach to this problem appears in [6]). Nevertheless, as mentioned in Section 3, a meromorphic extension of $\int_0^T J_z(x,t;x,t;\alpha)dt$, with T small enough can be performed and its finite part at $z = -1$ turns to be integrable in the tangential variables. A similar result holds, *a fortiori*, for $\int_0^T t^n J_z(x,t;x,t;\alpha)dt$, with $n = 1,2,3...$ Then, near the boundary, the Taylor expansion of the function $A_\alpha = \frac{d}{d\alpha}D(\alpha)_B$ will naturally appear, and the limits to be compared are $F.P. \lim\limits_{z\to -1}\{\lim\limits_{\bar{y}\to\bar{x}} \int_0^T t^n J_z(x,t;y,s;\alpha)dt\}$ and $R.P. \lim\limits_{\bar{y}\to\bar{x}}\{\lim\limits_{z\to -1} \int_0^T t^n J_z(x,t;y,s;\alpha)dt\} = R.P. \lim\limits_{\bar{y}\to\bar{x}} \int_0^T t^n G_B(x,t;y,s;\alpha)dt$.

The starting point for this comparison will be to carry out asymptotic expansions and to analyze the terms for which the iterated limits do not coincide (or do not even exist).

An expansion of $G_B(x,t,y,s)$ in $M\backslash\partial M$ in homogeneous and logarithmic functions of $(\bar{x}-\bar{y})$ can be obtained from (41) for $\lambda = 0$:

$$G_B(x,t,y,s) = \sum_{j=1-\nu}^{0} h_j(x,t,x-y,t-s) + M(x,t)\log|(x,t)-(y,s)| \\ +R(x,t,y,s), \tag{58}$$

with h_j the Fourier transform $\mathcal{F}^{-1}(c_{-\nu-j})$ of $c_{-\nu-j}$ for $j > 0$ and $h_0 = \mathcal{F}^{-1}(c_{-\nu}) - M(x,t)\log|(x,t)-(y,s)|$. The function $M(x,t)$ will be explicitly computed below (see (74)). Our convention for the Fourier transform

is

$$\mathcal{F}(f)(\bar{\xi}) = \hat{f}(\bar{\xi}) = \int f(\bar{x})\ e^{-i\bar{x}.\bar{\xi}}\ d\bar{x},$$
$$\mathcal{F}^{-1}(\hat{f})(\bar{x}) = f(\bar{x}) = \frac{1}{(2\pi)^{\nu}} \int \hat{f}(\bar{\xi})\ e^{i\bar{x}.\bar{\xi}}\ d\bar{\xi}. \tag{59}$$

For $t > 0$, $R(x,t,y,s)$ is continuous even at the diagonal $(y,s) = (x,t)$. Nevertheless, $R(x,t,y,s)|_{(y,s)=(x,t)}$is not integrable because of its singularities at $t = 0$. On the other hand, the functions $t^n R(x,t,y,t)$ are integrable with respect to the variable t for $y \neq x$ and $n = 0,1,2,....$An expansion of $\int_0^{\infty} t^n R(x,t,y,t)dt$ in homogeneous and logarithmic functions of $(x-y)$ can also be obtained from (58):

$$\int_0^{\infty} t^n R(x,t,y,t)dt = \sum_{j=n+2-\nu}^{0} g_{j,n}(x,x-y) + M_n(x)\log(|x-y|) + R_n(x,y) \tag{60}$$

where $R_n(x,y)$ is continuous even at $y = x$, and $g_{j,n}$ is the Fourier transform of the (homogeneous extension of) $\int_0^{\infty} t^n \tilde{d}_{-1-j}(x,t,\xi,t,0)dt$, with

$$\tilde{d}_{-1-j}(x,t,\xi,s,\lambda) = -\int_{\Gamma^-} e^{-is\tau}\ d_{-1-j}(x,t,\xi,\tau,\lambda)\ d\tau \tag{61}$$

for Γ^- a closed path enclosing the poles of $d_{-1-j}(x,t,\xi,\tau,\lambda)$ lying in $\{Im\ \tau > 0\}$.

Since $\tilde{d}_{-1-j}$ is homogeneous of degree $-j$ in (1/t, $\xi, 1/s, \lambda$), $g_{j,n}$ turns out to be homogeneous of degree j in $x-y$.

The following technical lemma will be used for the proof of Theorem 1:

Lemma 1: *Let $a(\xi)$ a function defined on $\mathbf{R}^{\nu}$, homogeneous of degree -ν for $|\xi| \geq 1$ and $a(\xi) = 0$ for $|\xi| < 1$. Then its Fourier transform can be written as*

$$\mathcal{F}^{-1}(a(\xi))(z) = h(z) +\ M\ \frac{\Omega_{\nu}}{(2\pi)^{\nu}}\ (\log|z|^{-1} + \mathcal{K}_{\nu}) + R(z), \tag{62}$$

where

a) $h(z)$ is a homogeneous function of degree 0, such that $\int_{|z|=1} h(z)\ d\sigma_z = 0$. It is given by

$$h(z) =\ \ \mathcal{F}^{-1}(P.V.[a(\xi/|\xi|) - M\ ]|\xi|^{-\nu})(z). \tag{63}$$

b)
$$M = \frac{1}{\Omega_\nu} \int_{|\xi|=1} a(\xi)\ d\sigma_\xi, \tag{64}$$
where $\Omega_\nu = Area(S^{\nu-1})$, *and* $\mathcal{K}_\nu = \ln 2 - \frac{1}{2}\gamma + \frac{1}{2}\frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)}$ *with* γ *the Euler's constant.*

c) $R(z)$ *is a function regular at* $z = 0$ *with* $R(0) = 0$.

Proof: we can decompose the function $a(\xi)$ as
$$a(\xi) = \tilde{a}(\xi) + M\ |\xi|^{-\nu}\chi(\xi), \tag{65}$$
where $\tilde{a}(\xi)$ is homogeneous of degree $-\nu$ and $\int_{|\xi|=1} \tilde{a}(\xi)\ d\sigma_\xi = 0$, and $\chi(\xi)$ is the characteristic function of $\{|\xi| \geq 1\}$.

Hence
$$\mathcal{F}^{-1}(\tilde{a}(\xi))(z) = h(z) - r_1(z), \tag{66}$$
where $h(z)$ is the Fourier transform of the distribution
$$S = P.V.[a(\xi/|\xi|) - M\ ]|\xi|^{-\nu}, \tag{67}$$
and $r_1(z)$ is the Fourier transform of the compactly supported distribution
$$P.V.[\tilde{a}(\xi/|\xi|)|\xi|^{-\nu}(1 - \chi(\xi))]. \tag{68}$$
Then
$$r_1(z) = \lim_{\epsilon\to 0} \int_{\epsilon\leq|\xi|\leq 1} \tilde{a}(\xi/|\xi|)|\xi|^{-\nu}\ e^{i\xi.z}\ \frac{d\xi}{(2\pi)^{\nu-1}} \tag{69}$$
is a function regular at $z = 0$, with $r_1(0) = 0$. As the distribution S coincides in $\mathbf{R}^\nu - \{0\}$ with a smooth function of degree $-\nu$, $h = \mathcal{F}^{-1}(S)$ is also a smooth homogeneous function of degree 0 and it can be seen that its mean vanishes in $S^{\nu-1}$.

On the other hand, a direct calculation gives
$$\begin{aligned}
&\mathcal{F}^{-1}(|\xi|^{-\nu}\chi(\xi))(z) \\
&= \frac{1}{(2\pi)^\nu} \int_1^\infty \frac{d\rho}{\rho}\Omega_{\nu-1} \int_0^\pi d\theta\ \sin^{\nu-2}(\theta)\ e^{i\rho|z|\cos\theta} \\
&= (2\pi)^{-\nu/2} \int_{|z|}^\infty d\rho\ \rho^{-\nu/2} J_{\frac{\nu}{2}-1}(\rho) \\
&= \frac{\Omega_\nu}{(2\pi)^\nu}\ \{\log|z|^{-1} + \mathcal{K}_\nu\},
\end{aligned} \tag{70}$$

where

$$\mathcal{K}_\nu = \int_0^1 d\rho\ \rho^{-\nu/2}\left[J_{\frac{\nu}{2}-1}(\rho) - \frac{\rho^{\frac{\nu}{2}-1}}{2^{\frac{\nu}{2}-1}\Gamma(\nu/2)}\right] + \int_1^\infty d\rho\ \rho^{-\nu/2} J_{\frac{\nu}{2}-1}(\rho)$$
$$= \ln 2 - \frac{1}{2}\gamma + \frac{1}{2}\frac{\Gamma'(\nu/2)}{\Gamma(\nu/2)} \tag{71}$$

with γ the Euler's constant ($\gamma = 0.5772...$). ∎

Now, we introduce the main result of this section.

Theorem 1: *Let M be a compact connected sub manifold of $\mathbf{R}^\nu$ of codimension zero with smooth boundary ∂M and E a k-dimensional complex vector bundle over M.*

Let $(D_\alpha)_B$ be a family of elliptic differential operators of first order, acting on the sections of E, with a fixed local boundary condition B on ∂M, and denote by $J_z(x,t;x,t;\alpha)$ the meromorphic extension of the evaluation at the diagonal of the kernel of $((D_\alpha)_B)^z$.

Let us assume that, for each α, $(D_\alpha)_B$ is invertible, the family is differentiable with respect to α, and $\frac{\partial}{\partial\alpha}(D_\alpha)_B f = A_\alpha f$, with A_α a differentiable function.

If V is a neighborhood of ∂M defined by $t < \epsilon$ and $T > 0$ small enough, then:

a)

$$\frac{\partial}{\partial\alpha}\ln\ Det(D_\alpha)_B$$

$$= \underset{z=-1}{F.P.}\left[\int_{\partial M}\int_0^T tr\left\{A_\alpha(x,t)\ J_z(x,t;x,t;\alpha)\ \right\} dt dx\right] \tag{72}$$

$$+ \underset{z=-1}{F.P.}\left[\int_{M/V} tr\left\{A_\alpha(\bar{x})\ J_z(\bar{x};\bar{x};\alpha)\ \right\} d\bar{x}\right],$$

where a suitable partition of the unity is understood. (This expression must be considered as the evaluation at $z = -1$ of the analytic extension).

b) For every α, the integral $\int_0^T A_\alpha(x,t)\ J_z(x,t;x,t;\alpha)dt$ is a meromorphic function of z, for each $x \in \partial M$, with a simple pole at $z=-1$. Its finite part (dropping, from now on, the index α for the sake of simplicity) is given by

$$\underset{z=-1}{F.P.}\ \int_0^T A(x,t)\ J_z(x,t;x,t)dt$$

$$= -\int_0^T A(x,t) \int_{|(\xi,\tau)|=1} \frac{i}{2\pi} \int_\Gamma \frac{\ln\lambda}{\lambda}\ c_{-\nu}(x,t;\xi,\tau;\lambda)\ d\lambda\ \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu}\ dt$$

$$+\sum_{l=0}^{\nu-2} \frac{\partial_t^l A(x,0)}{l!} \int_{|\xi|=1} \int_0^\infty t^l\ \frac{i}{2\pi} \int_\Gamma \frac{\ln\lambda}{\lambda}\ \tilde{d}_{-(\nu-1)+l}(x,t;\xi,t;\lambda)\ d\lambda\ \ dt\ \frac{d\sigma_\xi}{(2\pi)^{\nu-1}}$$

$$+\lim_{y\to x} \left\{ \int_0^T A(x,t) \left[G_B(x,t;y,t) - \sum_{l=1-\nu}^{0} h_l(x,t;x-y,0) \right.\right.$$

$$\left. -M(x,t)\ \frac{\Omega_\nu}{(2\pi)^\nu} \left(\ln|x-y|^{-1} + \mathcal{K}_\nu\right) \right] dt$$

$$+\sum_{j=0}^{\nu-2} \sum_{l=0}^{\nu-2-j} \frac{\partial_t^l A(x,0)}{l!}\ g_{j,l-(\nu-2-j)}(x,x-y)$$

$$\left. +\sum_{l=0}^{\nu-2} \frac{\partial_t^l A(x,0)}{l!}\ M_{\nu-2-l}(x)\ \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}} \left(\ln|x-y|^{-1} + \mathcal{K}_{\nu-1}\right) \right\}, \tag{73}$$

with

$$M(x,t) = \frac{1}{\Omega_\nu} \int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0)\ \ d\sigma_{\xi,\tau}$$

$$M_j(x) = \frac{1}{\Omega_{\nu-1}} \int_{|\xi|=1} \int_0^\infty t^{\nu-2-j}\ \ \tilde{d}_{-1-j}(x,t;\xi,t;0)\ dt\ d\sigma_\xi, \tag{74}$$

where $\Omega_n = Area(S^{n-1})$, and where h_l and g_l are related to the Green function

G_B as in (58) and (60)

$$h_{1-\nu+j}(x,t;w,u)=\mathcal{F}^{-1}_{(\xi,\tau)}\ \left[c_{-1-j}(x,t;(\xi,\tau)/|(\xi,\tau)|;0)\ |(\xi,\tau)|^{-1-j}\right](w,u),$$

$$h_0(x,t;w,u)=\mathcal{F}^{-1}_{(\xi,\tau)}\left[P.V.\left\{(c_{-\nu}(x,t;(\xi,\tau)/|(\xi,\tau)|;0)-M(x,t))\ \ |(\xi,\tau)|^{-\nu}\right\}\right](w,u),$$

$$g_{j,l}(x,w)=\mathcal{F}^{-1}_{\xi}\left[\int_0^\infty t^n\ \tilde{d}_{-1-j}(x,t;\xi/|\xi|,t;0)\ dt|\xi|^{-1-j-n}\right](w),$$

$$g_{j,0}(x,w)=\mathcal{F}^{-1}_{\xi}\left[P.V.\left[\int_0^\infty t^{\nu-j-2}\ \tilde{d}_{-1-j}(x,t;\xi/|\xi|,t;0)\ dt-M_j(x)\right]|\xi|^{-(\nu-1)}\right](w). \tag{75}$$

c) The integral $\int_{M\setminus V} tr\,[A_\alpha(\bar{x})\ J_z(\bar{x};\bar{x})]\ \ d\bar{x}$ in the second term in the r.h.s. of (72) , is a meromorphic function of z with a simple pole at $z=-1$. Its finite part is given by

$$\underset{z=-1}{F.P.}\ \int_{M\setminus V} tr\,[A_\alpha(\bar{x})\ J_z(\bar{x};\bar{x})]\ \ d\bar{x}$$

$$\begin{aligned}&=\int_{M\setminus V} A_\alpha(\bar{x})\int_{|\bar{\xi}|=1}\frac{i}{2\pi}\int\frac{\ln\lambda}{\lambda}\ c_{-\nu}(\bar{x},\bar{\xi};\lambda)\ d\lambda\frac{d\bar{\xi}}{(2\pi)^\nu}\\&+\int_{M\setminus V}\lim_{\bar{y}\to\bar{x}}\ A_\alpha(\bar{x})[G_B(\bar{x},\bar{y})-\sum_{l=1-\nu}^{0}h_l(\bar{x},\bar{x}-\bar{y})\\&-M(\bar{x})\frac{\Omega_\nu}{(2\pi)^\nu}(\ln|\bar{x}-\bar{y}|^{-1}+\mathcal{K}_\nu)]\ d\bar{x}.\end{aligned} \tag{76}$$

Proof: Statement a) is a direct consequence of (53), (55) and (56).

For proving b), we first establish a technical result obtained from the fundamental estimate

$$|t^n\partial_\xi^\alpha\tilde{d}_{-1-j}(x,t,\xi,s;\lambda)|\leq Ce^{-c(t+s)(|\xi|+|\lambda|)}(|\xi|+|\lambda|)^{-j-n-|\alpha|}, \tag{77}$$

for $t,s>0$, $\lambda\in\Lambda$, due to R.T. Seeley [17]:

Lemma 2:
Let us define

$$D_{-1-j}(x,t;\xi,t;z) \ \equiv \frac{i}{2\pi}\int_{\Gamma}\lambda^z\theta_1(\xi,\lambda)\ \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)\ d\lambda, \tag{78}$$

then

i) If $r(x,t)$ is a function satisfying $|r(x,t)|\leq Ct^n$ for $0<t<T$, $n\in\mathbf{N}$, $T>0$,

$$\int_0^T r(x,t)\int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;z)\ e^{i(x-y)\xi}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt \tag{79}$$

is an absolutely convergent integral for $Re(z)<j+n-\nu+1$. As a consequence, it is an analytic function of z in this region, and it is continuous in all the variables (x,y,z).

ii) If $x\neq y$, (79) is an absolutely convergent integral for all $z\in\mathbf{C}$, and so no analytic extension is needed out of the diagonal.

iii)

$$\int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z)\ \ dt \tag{80}$$

is an homogeneous function of ξ for $|\xi|\geq 1$, of degree $z-j-n$, analytic in z for $Re(z)<j+n$ and then

$$\begin{aligned}\int_{\mathbf{R}^{\nu-1}}\int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z)\ \ dt\ \frac{d\xi}{(2\pi)^{\nu-1}} \\ = \alpha_j^n(x;z)+\frac{1}{z-j-n+\nu-1}\beta_j^n(x;z)\end{aligned} \tag{81}$$

with

$$\begin{aligned}\alpha_j^n(x;z) &= \int_{|\xi|\leq 1}\int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z)\ \ dt\ \frac{d\xi}{(2\pi)^{\nu-1}} \\ \beta_j^n(x;z) &= \int_{|\xi|=1}\int_0^\infty t^n D_{-1-j}(x,t;\xi,t;z)\ \ dt\ \frac{d\xi}{(2\pi)^{\nu-1}}\end{aligned} \tag{82}$$

analytic functions of z for $Re(z)<j+n$.

iv)

$$\int_{\mathbf{R}^{\nu-1}}\int_T^\infty t^n D_{-1-j}(x,t;\xi,t;z)\ e^{i(x-y)\xi}\ dt\ \frac{d\xi}{(2\pi)^{\nu-1}} \tag{83}$$

is an entire function of z, continuous in (x,y,z).

Proof:

i) $\tilde{d}_{-1-j}(x,t;\xi,s;\lambda)$ is a homogeneous function of degree $-j$ in the variables $(\xi,t^{-1},s^{-1},\lambda)$ [17]. Then

$$D_{-1-j}(x,t;\xi,t;z) \ = |\xi|^{z-j+1} D_{-1-j}(x,|\xi|t;\xi/|\xi|,|\xi|t;z) \tag{84}$$

for $|\xi| \geq 1$. In fact,

$$\begin{aligned} D_{-1-j}(x,t;\xi,t;z) \ &= \frac{i}{2\pi}\int_\Gamma \lambda^z \theta_1(\xi,\lambda)\ \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)\ d\lambda \\ &= |\xi|^{-j}\frac{i}{2\pi}\int_\Gamma \lambda^z\ \tilde{d}_{-1-j}(x,|\xi|t;\xi/|\xi|,|\xi|t;\lambda/|\xi|)\ d\lambda \end{aligned} \tag{85}$$

because $\theta_1(\xi,\lambda)=1$ for $|\xi|\geq 1$; and taking $\mu=\lambda/|\xi|$, we get

$$|\xi|^{z-j+1}\frac{i}{2\pi}\int_{\Gamma_\xi} \mu^z\ \tilde{d}_{-1-j}(x,|\xi|t;\xi/|\xi|,|\xi|t;\mu)\ d\mu, \tag{86}$$

where $\Gamma_\xi = \{\lambda/|\xi| : \lambda\in\Gamma\}$. Since $\tilde{d}_{-1-j}(x,|\xi|t;\xi/|\xi|,|\xi|t;\mu)$ has no poles between the paths Γ_ξ and Γ for $|\xi|\geq 1$ [17], one can take $\int_\Gamma d\lambda$ in (86), thus obtaining (84).

For proving i), it is sufficient to see that the integrand in (79) is dominated by an absolutely integrable function.

We write

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^{\nu-1}} |r(x,t) D_{-1-j}(x,t;\xi,t;z)| \ \ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt \\ &\leq C\int_0^T \int_{|\xi|\leq 1} t^n |D_{-1-j}(x,t;\xi,t;z)| \ \ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt \\ &+C\int_0^T \int_{|\xi|\geq 1} t^n |D_{-1-j}(x,t;\xi,t;z)| \ \ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt. \end{aligned} \tag{87}$$

For the first integral we have

$$\begin{aligned} &\int_0^T \int_{|\xi|\leq 1} t^n |D_{-1-j}(x,t;\xi,t;z)| \ \ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt \\ &\leq \frac{1}{2\pi}\int_0^T \int_{|\xi|\leq 1} t^n \ \int_\Gamma |\lambda^z|\theta_1(\xi,\lambda)\ |\tilde{d}_{-1-j}(x,t;\xi,t;\lambda)|\ |d\lambda| \frac{d\xi}{(2\pi)^{\nu-1}}\ dt \end{aligned} \tag{88}$$

and, using the estimate (77), we can dominate it by

$$C \int_{\Gamma} |\lambda^z| (|\xi| + |\lambda|)^{-j-n} |d\lambda| \leq C \int_{\Gamma} |\lambda|^{Re(z)-j-n} |d\lambda|, \tag{89}$$

which is finite for $Re(z) < j + n - 1$.

For the second integral in (87) we get, from (84),

$$\int_0^T \int_{|\xi|\geq 1} t^n |D_{-1-j}(x,t;\xi,t;z)| \; \frac{d\xi}{(2\pi)^{\nu-1}} \; dt$$

$$= \frac{1}{2\pi} \int_0^T \int_{|\xi|\geq 1} t^n \; |\xi|^{Re(z)-j+1-n} \tag{90}$$

$$\times | \int_{\Gamma} \lambda^z \; (|\xi| t)^n \; \tilde{d}_{-1-j}(x, |\xi| t; \xi/|\xi|, |\xi| t; \lambda/|\xi|) \; d\lambda | \; \frac{d\xi}{(2\pi)^{\nu-1}} \; dt \; .$$

From Seeley's estimate (77), it is dominated by

$$C \int_0^T \int_{|\xi|\geq 1} \; |\xi|^{Re(z)-j+1-n} \int_{\Gamma} |\lambda^z| \; e^{-c|\xi| t (1+|\lambda|)} \; (1+|\lambda|)^{-j-n} |d\lambda| \; \frac{d\xi}{(2\pi)^{\nu-1}} \; dt \; . \tag{91}$$

By performing the integral in t, one gets

$$C \left(\int_{|\xi|\geq 1} \; |\xi|^{Re(z)-j-n} \; \frac{d\xi}{(2\pi)^{\nu-1}} \right) \left(\int_{\Gamma} |\lambda^z| \;\; (1+|\lambda|)^{-1-j-n} |d\lambda| \; \right), \tag{92}$$

which is finite for $Re(z) < j + n - \nu + 1$.

ii) We have, for $Re(z) < j + n - \nu + 1$

$$\int_0^T \int_{\mathbf{R}^{\nu-1}} r(x,t) \; D_{-1-j}(x,t;\xi,t;z) \; e^{i(x-y)\xi} \; \frac{d\xi}{(2\pi)^{\nu-1}} \; dt$$

$$= \int_{\mathbf{R}^{\nu-1}} \int_0^T r(x,t) \; D_{-1-j}(x,t;\xi,t;z) \; e^{i(x-y)\xi} \;\; dt \; \frac{d\xi}{(2\pi)^{\nu-1}}, \tag{93}$$

from part i).

Now, the estimate

$$\int_0^T |r(x,t)| \quad \int_\Gamma |\lambda^z| \; |\partial_\xi^\alpha [\theta_1(\xi,\lambda)\; \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)]| \; |d\lambda| dt$$

$$\leq C \int_0^T \int_\Gamma |\lambda^z| \; t^n \sum_{\alpha_1 \leq \alpha} C_{\alpha_1} |\partial_\xi^{\alpha_1} \theta_1(\xi,\lambda)| \; | \; \partial_\xi^{\alpha-\alpha_1} \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)| \; |d\lambda| dt$$

$$\leq C \int_\Gamma |\lambda^z| \int_0^T \; e^{-ct(|\xi|+|\lambda|)} (|\xi|+|\lambda|)^{-j-n-|\alpha|} \; dt |d\lambda|$$

$$= C \int_\Gamma |\lambda^z| \; (|\xi|+|\lambda|)^{-1-j-n-|\alpha|} \; dt |d\lambda|$$

$$\leq C \; (|\xi|+\epsilon)^{-\delta} \; \left(\int_\Gamma |\lambda^z| \; |\lambda|^{\delta-1-j-n-|\alpha|} \; |d\lambda| \right), \tag{94}$$

where $\delta > 0$ can be chosen such that the integral in λ is convergent for $Re(z) < 0$, implies

$$\partial_\xi^\alpha \left(\int_0^T r(x,t) \; D_{-1-j}(x,t;\xi,t;z) \;\; dt \right) \underset{\xi\to\infty}{\longrightarrow} 0 \;\; \text{for Re(z)} < 0 \text{ and } |\alpha| \geq 0. \tag{95}$$

Then, for $x \neq y$, writing $e^{i(x-y)\xi} = \frac{(-1)^k}{|x-y|^{2k}} \triangle_\xi^k e^{i(x-y)\xi}$, where $\triangle_\xi$ is the Laplacian in the ξ-variables, and integrating by parts in ξ, one gets that (93) becomes

$$\frac{(-1)^k}{|x-y|^{2k}} \int_{\mathbf{R}^{\nu-1}} \triangle_\xi^k \left(\int_0^T r(x,t) \; D_{-1-j}(x,t;\xi,t;z) \;\; dt \right) e^{i(x-y)\xi} \; \frac{d\xi}{(2\pi)^{\nu-1}}. \tag{96}$$

So, the expression in (96) defines a holomorphic function of z for $Re(z)$ as large as we want, taking k sufficiently large, since

$$\int_{\mathbf{R}^{\nu-1}} \int_0^T |r(x,t)| \, |\triangle_\xi^k \, D_{-1-j}(x,t;\xi,t;z)| \;\; dt \; d\xi$$

$$\leq C \int_{|\xi|\geq 1} \int_\Gamma |\lambda^z| \, |\triangle_\xi^k \;\; \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)| \, |d\lambda| \; d\xi$$

$$+C \int_{|\xi|\leq 1} \int_{\Gamma\cap\{|\lambda|\geq 1\}} |\lambda^z| \, |\triangle_\xi^k \;\; \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)| \, |d\lambda| \; d\xi \tag{97}$$

$$+C \int_{|\xi|\leq 1} \int_{\Gamma\cap\{|\lambda|\leq 1\}} |\lambda^z| \, |\triangle_\xi^k \; [\theta_1(\xi,\lambda) \; \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)]| \, |d\lambda| \; d\xi,$$

where the estimate $|\triangle_\xi^k \;\; \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)| \leq C \; (|\xi| + |\lambda|)^{-j-2k}$ guarantees the convergence for large $Re(z)$ of the first two terms, and the last one is convergent for all z.

iii) The integral $\int_0^T t^n \; D_{-1-j}(x,t;\xi,t;z) \;\; dt$ is absolutely convergent for $Re(z) < j+n$, as a consequence of the estimate (77). Its homogeneity is obvious, and then

$$\int_{\mathbf{R}^{\nu-1}} \int_0^\infty t^n \; D_{-1-j}(x,t;\xi,t;z) \;\; dt \; \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$= \int_{|\xi|\leq 1} \int_0^\infty t^n \; D_{-1-j}(x,t;\xi,t;z) \;\; dt \; \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$+ \int_{|\xi|=1} \left(\int_1^\infty r^{\nu-2+z-j-n} \; dr \right) \left(\int_0^\infty t^n \; D_{-1-j}(x,t;\xi,t;z) \;\; dt \right) \; \frac{d\sigma_\xi}{(2\pi)^{\nu-1}}$$

$$= \alpha_j^n(x;z) + \frac{1}{z-j-n+\nu-1} \beta_j^n(x;z), \tag{98}$$

for $Re(z) < j+n-\nu+1$.

It is easy to verify the analyticity of $\alpha_j^n(x;z)$ and $\beta_j^n(x;z)$ for $Re(z) < j+n$ by means of the estimate of $t^n \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)$.

iv) It is sufficient to prove that the integral is absolutely convergent for all z. In fact,

$$\int_{\mathbf{R}^{\nu-1}} \int_T^\infty t^n \int_\Gamma |\lambda^z| \theta_1(\xi,\lambda) \, |\tilde{d}_{-1-j}(x,t;\xi,t;\lambda)| \, |d\lambda| dt \, \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$\leq \int_{\mathbf{R}^{\nu-1}} \int_\Gamma |\lambda^z| \left(\int_T^\infty e^{-ct(|\xi|+|\lambda|)} \, dt \right) (|\xi| + |\lambda|)^{-j-n} \, \theta_1(\xi,\lambda) |d\lambda| dt \, \frac{d\xi}{(2\pi)^{\nu-1}} \, . \tag{99}$$

The integral in t is $\dfrac{e^{-cT(|\xi|+|\lambda|)}}{c(|\xi|+|\lambda|)}$ and $(|\xi| + |\lambda|)^{-1-j-n} \leq C$ in the support of $\theta_1(\xi,\lambda)$. Then, we can bound (99) by

$$C \left(\int_{\mathbf{R}^{\nu-1}} e^{-cT|\xi|} \frac{d\xi}{(2\pi)^{\nu-1}} \right) \left(\int_\Gamma |\lambda^z| \, e^{-cT|\lambda|} |d\lambda| \right) \tag{100}$$

which is finite for all values of z. ∎

In what follows, we will proof the assertion in (**??**). As a byproduct we will also obtain the following expression for the residue

$$\begin{aligned} &\operatorname*{Res}_{z=-1} \; \textstyle\int_0^T A(x,t) \; J_z(x,t;x,t) dt \\ &= - \int_0^T A(x,t) \int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0) \, \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} \, dt \\ &+ \sum_{l=0}^{\nu-2} \frac{\partial_t^l A(x,0)}{l!} \int_{|\xi|=1} \int_0^\infty t^l \; \tilde{d}_{-(\nu-1)+l}(x,t;\xi,t;0) \, dt \, \frac{d\sigma_\xi}{(2\pi)^{\nu-1}} \, . \end{aligned} \tag{101}$$

We can use , as an approximation to $(D_B - \lambda)^{-1}$ [17], the parametrix

$$P_K(\lambda) = \sum_\varphi \psi \left[\sum_{j=0}^K Op(\theta_2 \; c_{-1-j}) - \sum_{j=0}^K Op'(\theta_1 \; d_{-1-j}) \right] \; \varphi, \tag{102}$$

where φ is a partition of the unity, $\psi \equiv 1$ in $Supp(\varphi)$,

$$\theta_2(\xi, \tau, \lambda) = \chi(|\xi|^2 + |\tau|^2 + |\lambda|^2)$$
$$\theta_1(\xi, \lambda) = \chi(|\xi|^2 + |\lambda|^2), \tag{103}$$

with [17]

$$\chi(t) = \begin{cases} 0 & t \leq 1/2 \\ 1 & t \geq 1 \end{cases}, \tag{104}$$

and

$$Op(\sigma)h(x,t) = \int \sigma(x, t; \xi, \tau)\ \hat{h}(\xi, \tau)\ e^{i(x\xi + t\tau)} \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{d\tau}{2\pi},$$
$$Op'(\sigma)h(x,t) = \int\int \tilde{\sigma}(x, t; \xi, s)\ \tilde{h}(\xi, s)\ e^{ix\xi} \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{ds}{2\pi}, \tag{105}$$

where $\hat{h}(\xi, \tau)$ is defined in (59) and

$$\tilde{h}(\xi, s) = \int\ h(x, s)\ e^{-ix\xi}\ dx. \tag{106}$$

Thus, one can approximate the kernel J_z of D_B^z by means of the kernel L_z^K of $\frac{i}{2\pi} \int_\Gamma \lambda^z\ P_K(\lambda)\ d\lambda$. We have

$$L_z^K(x, t; y, s) = \sum_\varphi\ \psi(x, t) \left[\ \sum_{j=0}^{K} \int_{\mathbf{R}^\nu} C_{-1-j}(x, t; \xi, \tau; z)\ e^{i[(x-y)\xi + (t-s)\tau]}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{d\tau}{2\pi} \right.$$
$$\left. - \sum_{j=0}^{K} \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x, t; \xi, \tau; z)\ e^{i(x-y)\xi}\ \frac{d\xi}{(2\pi)^{\nu-1}} \right]\ \varphi(y, s) \tag{107}$$

where we have called

$$C_{-1-j}(x,t;\xi,\tau;z) = \frac{i}{2\pi}\int_\Gamma \lambda^z\ \theta_2(\xi,\tau;\lambda)\ \ c_{-1-j}(x,t;\xi,\tau;\lambda)\ d\lambda, \tag{108}$$

and

$$D_{-1-j}(x,t;\xi,t;z) = \frac{i}{2\pi}\int_\Gamma \lambda^z\ \theta_1(\xi;\lambda)\quad \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)\ d\lambda, \tag{109}$$

as in (78). These expressions are, in fact, analytic functions of z for all complex z, since the singularities of $c_{-1-j}(\lambda)$ and $\tilde{d}_{-1-j}(\lambda)$ are in a compact set in the λ plane, for $(x,t;\xi,\tau)$ in a compact set.

Since $(D_B-\lambda)^{-1} - P_K(\lambda)$ has a continuous kernel of $O(\ |\lambda|^{\nu-K-1})$ for $\lambda \in \Lambda$ [17], it turns out that

$$R(x,t;y,s;z) = J_z(x,t;y,s) - L_z^K(x,t;y,s) \tag{110}$$

is a continuous function of x,t,y,s and z, and analytic in z for $Re(z)<0$, if $K \geq \nu$. Analyzing the last terms in L_z^K, we obtain that it is also true for $K=\nu-1$. From now on, we call $L_z = L_z^{\nu-1}$. Then

$$\lim_{z\to -1}\left[\lim_{(y,s)\to(x,t)}(J_z - L_z)\right] = \lim_{(y,s)\to(x,t)}\left[\lim_{z\to -1}(J_z - L_z)\right]. \tag{111}$$

Since

$$J_{-1}(x,t;y,s) = G_B(x,t;y,s), \quad \text{for } (x,t)\neq(y,s), \tag{112}$$

we have

$$\lim_{z\to -1}(J_z(x,t;x,t) - L_z(x,t;x,t)) = \lim_{(y,s)\to(x,t)}(G_B(x,t;y,s) - L_{-1}(x,t;y,s)). \tag{113}$$

As will be shown in Lemma 3 below, one can cancel some terms in the equality (113) by studying the singularities of $L_{-1}(x,t;y,s)$ at $(x,t)=(y,s)$, and those of $L_z(x,t;x,t)$ at $z=-1$.

Lemma 3 :

The following statement holds

$$\lim_{z\to -1}\Bigg[J_z(x,t;x,t)+\frac{1}{(z+1)}\int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0)\ \frac{d\sigma_{\xi,\tau}}{(2\pi)^{\nu}}$$

$$+\int_{|(\xi,\tau)|=1}\frac{i}{2\pi}\int_{\Gamma}\frac{\ln\lambda}{\lambda}c_{-\nu}(x,t;\xi,\tau;\lambda)\ d\lambda\ \frac{d\sigma_{\xi,\tau}}{(2\pi)^{\nu}}$$

$$+\sum_{j=0}^{\nu-1}\int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;z)\quad \frac{d\xi}{(2\pi)^{\nu-1}}\Bigg]$$

$$(114)$$

$$=\lim_{y\to x}\Bigg[G_B(x,t;y,t)-\sum_{l=1-\nu}^{0} h_l(x,t;x-y,0)$$

$$-\ M(x,t)\ \frac{\Omega_\nu}{(2\pi)^{\nu}}(\ln|x-y|^{-1}+\mathcal{K}_\nu)$$

$$+\sum_{j=0}^{\nu-1}\int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;-1)e^{i(x-y)\xi}\quad \frac{d\xi}{(2\pi)^{\nu-1}}\Bigg].$$

Proof:

The terms in $L_z(x,t;x,t)$ involving the coefficients C_{-1-j} can be written as:

$$\sum_{j=0}^{\nu-1} \int C_{-1-j}(x,t;\xi,\tau;z) \ \ \frac{d\xi}{(2\pi)^{\nu-1}} \, \frac{d\tau}{2\pi}$$

$$= \sum_{j=0}^{\nu-2} \int_{|(\xi,\tau)|\leq 1} C_{-1-j}(x,t;\xi,\tau;z) \ \ \frac{d\xi}{(2\pi)^{\nu-1}} \, \frac{d\tau}{2\pi}$$

$$+ \frac{-1}{z-j-\nu} \int_{|(\xi,\tau)|=1} C_{-1-j}(x,t;\xi,\tau;z) \ d\sigma_{\xi,\tau}$$

$$+ \int_{|(\xi,\tau)|\leq 1} C_{-\nu}(x,t;\xi,\tau;z) \ \ \frac{d\xi}{(2\pi)^{\nu-1}} \, \frac{d\tau}{2\pi} \tag{115}$$

$$+ \frac{-1}{(\ z+1)} \int_{|(\xi,\tau)|=1} C_{-\nu}(x,t;\xi,\tau;z)|_{z=-1} \ \frac{d\sigma_{\xi,\tau}}{(2\pi)^{\nu}}$$

$$-i \int_{|(\xi,\tau)|=1} \partial_z C_{-\nu}(x,t;\xi,\tau;z)|_{z=-1} \ \frac{d\sigma_{\xi,\tau}}{(2\pi)^{\nu}}$$

$$+O(z+1),$$

because of the homogeneity properties of the functions C_{-1-j} for $|(\xi,\tau)| \geq 1$, and their analyticity in the variable z.

Analogously, considering $L_{-1}(x,t;x,s)$ we get

$$\sum_{j=0}^{\nu-1} \int C_{-1-j}(x,t;\xi,\tau;-1)\ e^{i[(x-y)\xi+(t-s)\tau]}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{d\tau}{2\pi}$$

$$= \sum_{j=0}^{\nu-2} \Big\{ h_{1-\nu+j}(x,t;x-y,t-s)$$

$$+ \int_{|(\xi,\tau)|\le 1} C_{-1-j}(x,t;\xi,\tau;-1)\ \ e^{i[(x-y)\xi+(t-s)\tau]}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{d\tau}{2\pi}$$

$$- \int_{|(\xi,\tau)|\le 1} C_{-1-j}(x,t;(\xi,\tau)/|(\xi,\tau)|;-1)\ \ |(\xi,\tau)|^{-1-j}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{d\tau}{2\pi} \Big\}$$

$$+h_0(x,t;x-y,t-s) +\ M(x,t)\frac{\Omega_\nu}{(2\pi)^\nu}\left[\ln(|x-y|^2+|t-s|^2)^{-1/2}+\mathcal{K}_\nu\right]$$

$$+ \int_{|(\xi,\tau)|\le 1} C_{-\nu}(x,t;(\xi,\tau)/|(\xi,\tau)|;-1)\ \ e^{i[(x-y)\xi+(t-s)\tau]}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ \frac{d\tau}{2\pi}$$

$$+O((x-y,t-s)), \tag{116}$$

where h_l, $l>0$, are the homogeneous functions obtained by Fourier transforming $C_{-1-j}(x,t;\frac{(\xi,\tau)}{|(\xi,\tau)|};-1)\ |(\xi,\tau)|^{-1-j}$. Lemma 1 was applied for calculating the Fourier transform of $C_{-1-j}(x,t;\xi,\tau;-1)$.

Then, we can write (115) as

$$\begin{aligned} &\frac{-1}{(\ z+1)}\int_{|(\xi,\tau)|=1} C_{-\nu}(x,t;\xi,\tau;-1)\ \ \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} \\ &- \int_{|(\xi,\tau)|=1} \partial_z C_{-\nu}(x,t;\xi,\tau;-1)\ \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu} + R_1(x,t,z) + O(z+1) \end{aligned} \tag{117}$$

and expression (116) as

$$\sum_{j=0}^{\nu-1} h_{-(\nu-1)+j}(x,t;x-y,t-s)$$

$$+ M(x,t)\frac{\Omega_\nu}{(2\pi)^\nu}\left[\ln(|x-y|^2+|t-s|^2)^{-1/2}+\mathcal{K}_\nu\right] \tag{118}$$

$$+R_2(x,t,y,s)+O((x-y,t-s)).$$

Taking into account that

$$\lim_{z\to -1} R_1(x,t,z) = \lim_{(y,s)\to(x,t)} R_2(x,t,y,s), \tag{119}$$

and that

$$C_{-\nu}(x,t;\xi,\tau;-1) = c_{-\nu}(x,t;\xi,\tau;0) \quad \text{for} \quad |(\xi,\tau)| \geq 1$$

$$\partial_z C_{-\nu}(x,t;\xi,\tau;-1) = \frac{i}{2\pi}\int_\Gamma \frac{\ln\lambda}{\lambda}\, c_{-\nu}(x,t;\xi,\tau;\lambda)\, d\lambda, \tag{120}$$

we obtain Lemma 3. ∎

The meromorphic extension of the terms involving the coefficients $C_{-1-j}(x,t;\xi,\tau;z)$ in $L_z(x,t;x,t)$ is a consequence of expression (116). Although $\sum_{j=0}^{\nu-1}\int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,t;z)\ \frac{d\xi}{(2\pi)^{\nu-1}}$ does not admit, in general, a meromorphic extension, such extension can be performed for

$$\int_0^T t^n \int_{\mathbf{R}^{\nu-1}} D_{-1-j}(x,t;\xi,\tau;z)\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt, \tag{121}$$

for $n=0,1,...$ and $j=0,1,2,...$ (see [18] and Lemma 2).

In order to get part b) of the theorem, we study the limits $\lim_{z\to-1}\int_0^T A(x,t)\ R(x,t;z)\ dt$ and $\lim_{y\to x}\int_0^T A(x,t)\ S(x,t;y,t)\ dt$, where $R(x,t;z)$ and $S(x,t;y,t)$ denote the expressions appearing in the limits on the l.h.s. and r.h.s. of (114) respectively. (We have written $A(x,t)=A_\alpha(x,t)$ for notational simplicity).

Lemma 4: *If $A(x,t)$ has $\nu-1-j$ continuous derivatives in the variable t, $t\geq 0$, then*

i) For $\nu-1-j>0$,

$$\int_0^T A(x,t)\int D_{-1-j}(x,t;\xi,t;z)\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt=\psi_j(x,z)$$

$$-\frac{1}{z+1}\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\ \int_{|\xi|=1}\int_0^\infty t^{\nu-j-2}D_{-1-j}(x,t;\xi,t;-1)\ dt\ \frac{d\sigma_\xi}{(2\pi)^{\nu-1}}$$

$$-\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\int_{|\xi|=1}\int_0^\infty t^{\nu-j-2}\partial_z D_{-1-j}(x,t;\xi,t;-1)\ dt\frac{d\sigma_\xi}{(2\pi)^{\nu-1}}, \tag{122}$$

with $\psi_j(x,z)$ an analytic function of z for $Re(z)<0$.

Moreover,

$$\int_0^T A(x,t)\int D_{-1-j}(x,t;\xi,t;-1)\ e^{i(x-y)\xi}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt$$

$$=\varphi_j(x,y)\ +\sum_{\substack{n=0\\ l=j+n-\nu+2}}^{\nu-j-2}\frac{\partial_t^n A(x,0)}{n!}\ g_{j,l}(x,x-y) \tag{123}$$

$$+\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\quad M_j(x)\ \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}}(\ln|x-y|^{-1}+\mathcal{K}_{\nu-1}),$$

where $\varphi_j(x,y)$ is a continuous function.

ii) For $\nu-1-j=0$,

$$\int_0^T A(x,t)\int D_{-1-j}(x,t;\xi,t;z)\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt=\psi_j(x,z) \tag{124}$$

is an analytic function of z for $Re(z)<0$, and

$$\int_0^T A(x,t)\int D_{-1-j}(x,t;\xi,t;-1)\ e^{i(x-y)\xi}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt=\varphi_j(x,y) \tag{125}$$

is a continuous function.

iii) For all j

$$\lim_{z\to -1} \ \psi_j(x,z) = \lim_{y\to x} \ \varphi_j(x,y) \tag{126}$$

Proof:
For $\nu - 1 - j \geq 0$, let

$$A(x,t) = \sum_{n=0}^{\nu-j-2} \frac{\partial_t^n A(x,0)}{n!} \ t^n \ + \ \epsilon_j(x,t) \ t^{\nu-1-j}, \tag{127}$$

with $|\epsilon_j(x,t)| \leq C$, for $t \in [0,T]$. Then, we write

$$\begin{aligned}
&\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;z) \ \frac{d\xi}{(2\pi)^{\nu-1}} \ dt \\
&= \int \left(\int_0^T A(x,t) D_{-1-j}(x,t;\xi,t;z) \ \ dt \right) \ \frac{d\xi}{(2\pi)^{\nu-1}} \\
&= \sum_{n=0}^{\nu-j-2} \frac{\partial_t^n A(x,0)}{n!} \ \int \left(\int_0^T t^n \ D_{-1-j}(x,t;\xi,t;z) \ dt \right) \frac{d\xi}{(2\pi)^{\nu-1}} \\
&+ \int \left(\int_0^T \epsilon_j(x,t) \ t^{\nu-1-j} \ D_{-1-j}(x,t;\xi,t;z) \ dt \right) \frac{d\xi}{(2\pi)^{\nu-1}} \ ,
\end{aligned} \tag{128}$$

where the first equality holds for $Re(z) < j-\nu+1$ by the estimates in Lemma 2 i). Also, by Lemma 2 i) the last term is analytic for $Re(z) < 0$ and it can be written as

$$\int \int_0^T \epsilon_j(x,t) \ t^{\nu-1-j} \ D_{-1-j}(x,t;\xi,t;-1) \ dt \ \ \frac{d\xi}{(2\pi)^{\nu-1}} \ + \ O(z+1). \tag{129}$$

Finally, by Lemma 2 iv), and evaluating this analytic functions at $z = -1$,

expression (128) gives

$$\sum_{n=0}^{\nu-j-3} \frac{\partial_t^n A(x,0)}{n!} \left[\alpha_j^n(x;-1) + \frac{-1}{z-j-n+\nu-1} \beta_j^n(x;-1) \right.$$

$$\left. - \int \int_T^\infty t^n \ D_{-1-j}(x,t;\xi,t;-1) \ dt \ \frac{d\xi}{(2\pi)^{\nu-1}} \right]$$

$$+ \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \left[\alpha_j^{\nu-j-2}(x;-1) + \frac{-1}{(z+1)} \beta_j^{\nu-j-2}(x;-1) - \partial_z \beta_j^{\nu-j-2}(x;-1) \right.$$

$$\left. + \int \int_T^\infty t^{\nu-j-2} \ D_{-\nu+1}(x,t;\xi,t;-1) \ dt \ \frac{d\xi}{(2\pi)^{\nu-1}} \right]$$

$$+ \int \left(\int_0^T \epsilon_j(x,t) \ t^{\nu-j-2} \ D_{-1-j}(x,t;\xi,t;-1) \ dt \right) \frac{d\xi}{(2\pi)^{\nu-1}} + O(z+1). \tag{130}$$

On the other hand,

$$\int_0^T A(x,t) \quad \left(\int D_{-1-j}(x,t;\xi,t;-1) \ e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}} \right) \ dt$$
$$= \int \left(\int_0^T A(x,t) \quad \int D_{-1-j}(x,t;\xi,t;-1) \ dt \right) \ e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}} \tag{131}$$

since, for $x \neq y$, the integral is absolutely convergent (Lemma 2 ii)). Then, it can be written as

$$\sum_{n=0}^{\nu-j-2} \frac{\partial_t^n A(x,0)}{n!} \int \left(\int_0^\infty t^n \ D_{-1-j}(x,t;\xi,t;-1) \ dt \right) \ e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$- \sum_{n=0}^{\nu-j-2} \frac{\partial_t^n A(x,0)}{n!} \int \left(\int_T^\infty t^n \ D_{-1-j}(x,t;\xi,t;-1) \ dt \right) \ e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$+ \int \left(\int_0^T \epsilon_j(x,t) \ t^{\nu-1-j} \ D_{-1-j}(x,t;\xi,t;-1) dt \right) \ e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}}. \tag{132}$$

Here, the integral $\int_0^\infty t^n \ D_{-1-j}(x,t;\xi,t;-1)\ dt$ is a homogeneous function of degree $-1-j-n$ for $|\xi| \geq 1$. So, its Fourier transform evaluated at $x-y$ can, for $-1-j-n \geq -(\nu-2)$, be written as

$$g_{j,l}(x,x-y) + \int_{|\xi|\leq 1} \left(\int_0^\infty t^n \ D_{-1-j}(x,t;\xi,t;-1)\ dt \ \right) e^{i(x-y)\xi} \ \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$- \int_{|\xi|\leq 1} \left(\int_0^\infty t^n \ D_{-1-j}(x,t;\xi/|\xi|,t;-1)\ dt \right) |\xi|^{-1-j-n} e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}}$$

$$= g_{j,l}(x,x-y) + \alpha_j^n(x,-1) - \frac{1}{-j-n+\nu-2} \beta_j^n(x,-1) + O(|x-y|), \tag{133}$$

where $g_{j,l}(x,w)$ is the homogeneous function of degree $l = j+n-\nu+2$ defined in (75).

When $-1-j-n = -(\nu-1)$, by Lemma 1, we have that the Fourier transform of $\int_0^\infty t^{\nu-j-2} \ D_{-1-j}(x,t;\xi,t;-1)\ dt$ is given by

$$g_{j,0}(x,x-y) + \ M_j(x) \ \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}} (\ln|x-y|^{-1} + \mathcal{K}_{\nu-1}) + O(|x-y|)$$

$$+ \int_{|\xi|\leq 1} \left(\int_0^\infty t^{\nu-j-2} \ D_{-1-j}(x,t;\xi,t;-1)\ dt \right) e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}} \tag{134}$$

$$= g_{j,0}(x,x-y) + \ M_j(x) \ \ \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}} (\ln|x-y|^{-1} + \mathcal{K}_{\nu-1})$$

$$+ \alpha_j^{\nu-j-2}(x,-1) + O(|x-y|).$$

The terms

$$\frac{\partial_t^n A(x,0)}{n!} \int \left(\int_T^\infty t^n \ D_{-1-j}(x,t;\xi,t;-1)\ dt \right) \ e^{i(x-y)\xi} \frac{d\xi}{(2\pi)^{\nu-1}} \tag{135}$$

in (132) have a limit when $y \to x$. Then, from (131) to (134) we obtain

$$\int_0^T A(x,t) \int D_{-1-j}(x,t;\xi,t;-1)\ e^{i(x-y)\xi}\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt$$

$$= \sum_{n=0}^{\nu-j-3} \frac{\partial_t^n A(x,0)}{n!} g_{j,j+n-\nu+2}(x,x-y)$$

$$+ \sum_{n=0}^{\nu-j-3} \frac{\partial_t^n A(x,0)}{n!} \left[\alpha_j^n(x,-1) - \frac{1}{-j-n+\nu-2}\ \beta_j^n(x,-1)\right.$$

$$\left. - \int \int_T^\infty t^n\ D_{-1-j}(x,t;\xi,t;-1)\ dt\ \frac{d\xi}{(2\pi)^{\nu-1}}\right]$$

$$+ \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \left[g_{j,0}(x,x-j) + \quad M_j(x)\ \frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}}(\ln|x-y|^{-1} + \mathcal{K}_{\nu-1})\right]$$

$$+ \frac{\partial_t^{\nu-j-2} A(x,0)}{(\nu-j-2)!} \left[\alpha_j^{\nu-j-2}(x,-1) - \int\int_T^\infty t^{\nu-j-2}\ D_{-\nu+1}(x,t;\xi,t;-1)\ dt\ \frac{d\xi}{(2\pi)^{\nu-1}}\right]$$

$$+ \int \left(\int_0^T \epsilon_j(x,t)\ t^{\nu-1-j}\ D_{-1-j}(x,t;\xi,t;-1)\ dt\right)\ \frac{d\xi}{(2\pi)^{\nu-1}} + O(x-y), \tag{136}$$

and then, comparing expressions (47) and (136), Lemma 4 follows. ∎

Finally, in order to get part b) of Theorem 1 we write the equality in Lemma 3 as

$$\lim_{z\to -1}\ R(x,t;z) = \lim_{y\to x}\ S(x,y,t) \tag{137}$$

and evaluate the integrals $\int_0^T A(x,t)\ R(x,t;z)\ dt$ and $\int_0^T A(x,t)\ S(x,y,t)\ dt$.

For the first one, we have

$$\int_0^T A(x,t)\left[J_z(x,t;x,t)+\frac{1}{z+1}\int_{|(\xi,\tau)|=1} c_{-\nu}(x,t;\xi,\tau;0)\frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu}\right.$$

$$\left.+\int_{|(\xi,\tau)|=1}\frac{i}{2\pi}\int\frac{\ln\lambda}{\lambda}c_{-\nu}(x,t;\xi,\tau;\lambda)\ d\lambda\ \frac{d\sigma_{\xi,\tau}}{(2\pi)^\nu}\right]\ dt$$

$$=-\sum_{j=0}^{\nu-1}\int_0^T A(x,t)\ \int D_{-1-j}(x,t;\xi,t;z)\ \frac{d\xi}{(2\pi)^{\nu-1}}\ dt$$

$$+\int_0^T A(x,t)\ R(x,t;z)\ dt$$

$$=\sum_{j=0}^{\nu-2}\frac{1}{z+1}\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\int_{|\xi|=1}\int_0^\infty t^{\nu-j-2}D_{-1-j}(x,t;\xi,t;-1)\ \ dt\ \frac{d\sigma_\xi}{(2\pi)^{\nu-1}}$$

$$+\sum_{j=0}^{\nu-2}\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\int_{|\xi|=1}\int_0^\infty t^{\nu-j-2}\partial_z D_{-1-j}(x,t;\xi,t;-1)\ \ dt\ \frac{d\sigma_\xi}{(2\pi)^{\nu-1}}$$

$$-\sum_{j=0}^{\nu-1}\psi_j(x,z)+\int_0^T A(x,t)\ \ R(x,t;z)\ dt. \tag{138}$$

For the integral involving $S(x,y,t)$, we have

$$\int_0^T A(x,t)\,[G_B(x,t;y,t)$$

$$-\sum_{l=-(\nu-1)}^{0} h_l(x,t;x-y;0) - M(x,t)\frac{\Omega_\nu}{(2\pi)^\nu}\left(\ln|x-y|^{-1}+\mathcal{K}_\nu\right)\Big]\,dt$$

$$= -\sum_{j=0}^{\nu-1}\int_0^T A(x,t)\int D_{-1-j}(x,t;\xi,t;-1)\,e^{i(x-y)\xi}\,\frac{d\xi}{(2\pi)^{\nu-1}}\,dt$$

$$+\int_0^T A(x,t)\quad S(x,y,t)\,dt \tag{139}$$

$$= -\sum_{j=0}^{\nu-2}\sum_{n=0}^{\nu-j-2}\frac{\partial_t^n A(x,0)}{n!}g_{j,j+n+\nu-2}(x,x-y)$$

$$-\sum_{j=0}^{\nu-2}\frac{\partial_t^{\nu-j-2}A(x,0)}{(\nu-j-2)!}\quad M_j(x)\frac{\Omega_{\nu-1}}{(2\pi)^{\nu-1}}\left(\ln|x-y|^{-1}+\mathcal{K}_{\nu-1}\right)$$

$$-\sum_{j=0}^{\nu-1}\varphi_j(x,y)+\int_0^T A(x,t)\quad S(x,y,t)\,dt.$$

Then, taking into account that the last terms in (138) and (139) satisfy

$$\lim_{z\to-1}\left(-\sum_{j=0}^{\nu-1}\psi_j(x,z)+\int_0^T A(x,t)\quad R(x,t;z)\,dt\right)$$

$$= \lim_{y\to x}\left(-\sum_{j=0}^{\nu-1}\varphi_j(x,y)+\int_0^T A(x,t)\quad S(x,y,t)\,dt\right), \tag{140}$$

we obtain part b) of Theorem 1. Notice that, for $|\xi|\geq 1$,

$$D_{-1-j}(x,t;\xi,t;-1) = \quad \tilde{d}_{-1-j}(x,t;\xi,t;0) \tag{141}$$

and

$$\partial_z D_{-1-j}(x,t;\xi,t;-1) = \frac{i}{2\pi}\int_\Gamma \frac{\ln\lambda}{\lambda}\quad \tilde{d}_{-1-j}(x,t;\xi,t;\lambda)\,d\lambda. \tag{142}$$

The proof of c) is similar to the one of b), and even simpler because in this case the parametrix in (102) does not include terms of the form Op'(θ,d$_{-1-j}$). ∎

Awful as it looks, (**??**) is not so complicated: In the first place, all terms can be systematically evaluated. Moreover, the terms containing h_l subtract the singular part of the Green function in the interior of the manifold (see (58)) and can, thus, be easily identified from the knowledge of G_B. $R(x,t,y,t)$, the regular part so obtained, is still nonintegrable near the boundary. Those terms containing $g_{j,l}$ subtract the singular part of the integrals $\int_0^T\ t^n\ R(x,t,y,t)\ dt$ (see (60)). Finally, the terms containing $c_{-\nu}$ and $d_{-\nu+1}$ arise as a consequence of having replaced an analytic regularization by a *point splitting* one.

Even though Seeley's coefficients c and $\tilde{d}$ are to be obtained through an iterative procedure, which can make their evaluation a tedious task, in the cases of physical interest only the few first of them are needed. In fact, for the two dimensional example in the next section we will only need two such coefficients.

5 Two dimensional Dirac Operator on a disk.

In this section, we will use the method previously discussed to evaluate the determinant of the operator $D = i\partial\!\!\!/ + A\!\!\!/$ acting on functions defined on a two dimensional disk of radius R. A family of local bag-like [5] elliptic boundary conditions will be assumed.

We take A_μ to be an Abelian field in the Lorentz gauge; as it is well known, it can be written as $A_\mu = \epsilon_{\mu\nu}\ \partial_\nu\phi$ ($\epsilon_{01} = -\epsilon_{10} = 1$). For ϕ we choose a smooth bounded function $\phi = \phi(r)$. Notice that, with these assumptions, $A_r = 0$ and $A_\theta(r) = -\partial_r\phi(r)$.

We call

$$\Phi = \oint_{r=R} A_\theta\ R\ d\theta = -2\pi R\ \partial_r\phi(r)|_{r=R}. \tag{143}$$

Our convention for two dimensional Dirac matrices is as in (5):

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{144}$$

which satisfy:

$$\gamma_\mu \gamma_\nu \; = \delta_{\mu\nu} \; I \; + \; i \; \epsilon_{\mu\nu} \; \gamma_5. \tag{145}$$

Therefore, the free Dirac operator can be written as:

$$i \; \partial\!\!\!/ = i \; (\gamma_0 \; \partial_0 + \gamma_1 \; \partial_1) = 2i \; \begin{pmatrix} 0 & \frac{\partial}{\partial X} \\ \frac{\partial}{\partial X^*} & 0 \end{pmatrix}, \tag{146}$$

where $X = x_0 + i \; x_1$ and $X^* = x_0 - i \; x_1$.

Or, in polar coordinates:

$$i \; \partial\!\!\!/ = i(\gamma_r \; \partial_r + \frac{1}{r} \gamma_\theta \; \partial_\theta), \tag{147}$$

with

$$\gamma_r = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}, \qquad \gamma_\theta = \begin{pmatrix} 0 & -ie^{-i\theta} \\ ie^{i\theta} & 0 \end{pmatrix}. \tag{148}$$

With these conventions, the full Dirac operator can be written as:

$$D = e^{-\gamma_5 \phi(r)} \; i \; \partial\!\!\!/ \; e^{-\gamma_5 \phi(r)}. \tag{149}$$

Now, in order to perform our calculations, we consider the family of operators:

$$D_\alpha = i \; \partial\!\!\!/ + \alpha \; A\!\!\!/ = e^{-\alpha \gamma_5 \phi(r)} \; \; i \; \partial\!\!\!/ \; e^{-\alpha \gamma_5 \phi(r)}, \text{ with } 0 \leq \alpha \leq 1, \tag{150}$$

which will allow us to go smoothly from the free to the full Dirac operator. If we call

$$W(\alpha) = \ln \; Det(D_\alpha)_B, \tag{151}$$

where B represents the elliptic boundary condition, we have

$$\frac{\partial}{\partial \alpha} W(\alpha) = \underset{z=0}{F.P.} \; \left[Tr \left(A\!\!\!/ \; (D_\alpha)_B^{-z-1} \right) \right]. \tag{152}$$

From the Theorem in the previous section we get:

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}W(\alpha) = \frac{1}{(2\pi)^2}\; tr\Bigg\{\int \lim_{y\to x}\Bigg[\int \Big[\not{A}(t)\Big(4\pi^2 G_B(x,t,y,t) \\
-\frac{1}{|x-y|}\int\; e^{i\xi\frac{(x-y)}{|x-y|}}\; c_{-1}(x,t;\frac{(\xi,\tau)}{|(\xi,\tau)|};0)\; d\xi\; d\tau \\
-\int_{|(\xi,\tau)|\geq 1}\; e^{i\xi(x-y)}\; c_{-2}(x,t;\xi,\tau;0)\; d\xi\; d\tau \\
-\int\frac{i}{2\pi}\int_\Gamma\;\frac{\ln\lambda}{\lambda}\; c_{-2}(x,t;\frac{(\xi,\tau)}{|(\xi,\tau)|};\lambda)\; d\lambda\; d\sigma_{\xi,\tau}\Big) \\
+\not{A}(0)\Big(\int_{|\xi|\geq 1}\; e^{i\xi(x-y)}\; \tilde{d}_{-1}(x,t;\xi,t;0)\; d\xi \\
+\int\frac{i}{2\pi}\int_\Gamma\frac{\ln\lambda}{\lambda}\;\tilde{d}_{-1}(x,t;\frac{(\xi)}{|\xi|},t;\lambda)\; d\lambda\; d\sigma_\xi\Big)\Big]dt\Bigg]\, dx\Bigg\},
\end{aligned}
\tag{153}
$$

where the Fourier transforms of c_{-2} and $\tilde{d}_{-1}$ have been left explicitly indicated, instead of using the results of Lemma 1.

Now, the coefficients c and $\tilde{d}$ in the previous equation are those appearing in the asymptotic expansion of the resolvent $(D_\alpha - \lambda I\ )^{-1}$.

From (149), the symbol of $(D_\alpha - \lambda I)$ is:

$$
\begin{aligned}
\sigma(D_\alpha - \lambda I) &= (-\not{\xi} - \lambda I) + \alpha\not{A} \\
&= a_1(\theta,t,\xi,\tau,\lambda) + a_0(\theta,t,\xi,\tau,\lambda),
\end{aligned}
\tag{154}
$$

where

$$
\begin{aligned}
a_1 &= -\not{\xi} - \lambda I, \\
a_0 &= \alpha\not{A}.
\end{aligned}
\tag{155}
$$

The c -coefficients can then be obtained from (42) which, written in detail, gives

$$
\begin{aligned}
c_{-1} &= a_1^{-1} \\
a_1 c_{-1-j} + \sum\Big[(D_{\xi,\tau})^\beta a_k\Big]\,(iD_{x,t})^\beta\, c_{-1-l}/\beta! &= 0, \quad j = 1,2,...
\end{aligned}
\tag{156}
$$

where the sum is taken over all $l < j$ and $k - |\beta| - 1 - l = -j$.

So, the required Seeley's c-coefficients are given by [8] :

$$c_{-1} = \frac{1}{(\lambda^2 - \xi^2 - \tau^2)} (\not{\xi} - \lambda I),$$
$$c_{-2} = \frac{\alpha}{(\lambda^2 - \xi^2)^2} (2\lambda \xi_\mu A_\mu I - (\lambda^2 - \xi^2) \not{A} - 2\xi_\mu A_\mu \not{\xi}), \tag{157}$$

where $\not{\xi} = \xi\gamma_\theta + \tau\gamma_t$.

As regards the boundary contributors to the parametrix, i.e., the coefficients d_{-1-j}, they are the solutions of (44-47).

Now, from (46)

$$a^{(0)} = a_0|_{t=0} = \alpha \not{A}|_{r=R},$$
$$a^{(1)} = a_1|_{t=0} = -\lambda I - \xi\gamma_\theta + i\gamma_t \frac{\partial}{\partial t}. \tag{158}$$

In our case, the equation to be solved is

$$a^{(1)} d_{-1} = (-\lambda I - \xi\gamma_\theta + i\gamma_t \partial_t) d_{-1} = 0, \tag{159}$$

with boundary conditions

$$b_0 \ d_{-1} = b_0 \ c_{-1} \quad \text{at t} = 0, \tag{160}$$

plus the vanishing of d_{-1} as $t \to +\infty$. (159) can be recast in the form

$$\partial_t d_{-1} = -M \ d_{-1,} \tag{161}$$

where $M = \xi\gamma_5 + i\lambda\gamma_t$. It can be easily verified that

$$tr(M) = 0,$$
$$M^2 = (\xi^2 - \lambda^2) I. \tag{162}$$

So, M has eigenvalues $\pm\sqrt{\xi^2 - \lambda^2}$, corresponding to the eigenvectors

$$u_\pm = \begin{pmatrix} ie^{-i\theta}(\xi \pm \sqrt{\xi^2 - \lambda^2}) \\ \lambda \end{pmatrix}. \tag{163}$$

Now, the general solution to (161) is:

$$d_{-1}(x,t;\xi,\tau;\lambda) = e^{-tM}\ d_{-1}(x,0;\xi,\tau;\lambda). \tag{164}$$

Since $d_{-1} \to 0$ for $t \to \infty$, we get

$$d_{-1}(x,t;\xi,\tau;\lambda) = e^{-t\sqrt{\xi^2-\lambda^2}} u_+ \otimes \begin{pmatrix} f \\ g \end{pmatrix}^\dagger , \tag{165}$$

where the vector $\binom{f}{g}$ must be determined from the boundary condition at $t = 0$ $(r = R)$, given by (160). After this brief review of the general points, let us now go to the specific calculations.

We now consider a parametric family of bag-like local boundary conditions leading to an elliptic boundary problem. According to the discussion leading to (32), we choose the matrix

$$b_0 = \left(1, w\ e^{-i\theta}\right), \tag{166}$$

with w a nonzero complex constant.

We define the operator $(D_\alpha)_B$ as the differential operator in (149) , acting on the dense subspace of functions satisfying:

$$B\ \psi \equiv b_0 \psi|_{t=0} = 0. \tag{167}$$

Notice that these boundary conditions reduce to those of an MIT bag [5] when $w = \pm 1$.

5.1 Zero modes

We will here show that, with these boundary conditions, the operator is invertible. From (149), we get:

$$D_\alpha\ \psi = 0 \Rightarrow \not{\partial}\ e^{-\alpha\gamma_5\phi(r)}\ \psi = 0, \tag{168}$$

or, equivalently:

$$\begin{pmatrix} 0 & e^{-i\theta}(\partial_r - \frac{i}{r}\partial_\theta) \\ e^{i\theta}(\partial_r + \frac{i}{r}\partial_\theta) & 0 \end{pmatrix} \begin{pmatrix} e^{-\alpha\phi(r)} & 0 \\ 0 & e^{\alpha\phi(r)} \end{pmatrix} \begin{pmatrix} \varphi(r,\theta) \\ \chi(r,\theta) \end{pmatrix} = 0. \tag{169}$$

Now, we introduce the expansions:

$$
\begin{aligned}
\varphi(r,\theta) &= \sum_{n=-\infty}^{\infty} \varphi_n(r)\ e^{in\theta}, \\
\chi(r,\theta) &= \sum_{n=-\infty}^{\infty} \chi_n(r)\ e^{in\theta}.
\end{aligned}
\tag{170}
$$

The solutions are thus given by:

$$
\begin{aligned}
\varphi_n(r) &= a_n \ \ r^n \ e^{\alpha\phi(r)}, \\
\chi_n(r) &= b_n \ \ r^{-n} \ e^{-\alpha\phi(r)},
\end{aligned}
\tag{171}
$$

where the coefficients a_n and b_n are to be determined from the boundary conditions at $r = 0$ and $r = R$. The requirement of normalizability implies that :

$$
\begin{aligned}
a_n &= 0, \quad \text{for n} < 0, \\
b_n &= 0, \quad \text{for n} > 0.
\end{aligned}
\tag{172}
$$

At $r = R$, we get, from (167):

$$
b_{n+1} = -\frac{a_n}{w} \ R^{2n+1} \ e^{2\alpha\phi(R)}, \tag{173}
$$

which requires

$$
\begin{aligned}
a_n &= 0, \quad \text{for n} \geq 0, \\
b_n &= 0, \quad \text{for n} \leq 0.
\end{aligned}
\tag{174}
$$

So, for these local boundary conditions, there are no normalizable zero modes. Notice that these are not the most general local elliptic boundary conditions. In fact, zero modes would in general arise if one allowed w to depend on θ.

5.2 Computation of d_{-1}

Restricting ourselves to the case of θ-independent parameter w, we now go back to (165), from which:

$$
d_{-1}|_{r=R} = u_+ \otimes \begin{pmatrix} f \\ g \end{pmatrix}^{\dagger}, \tag{175}
$$

and determine $\binom{f}{g}^{\dagger}$ from the boundary condition:

$$b_0 \; d_{-1} = b_0 \; c_{-1}, \quad \text{at r = R}, \tag{176}$$

which, written in an explicit manner reads:

$$\{(1, w \; e^{-i\theta}) \; . \; u_+\} \begin{pmatrix} f \\ g \end{pmatrix}^{\dagger} = (1, w \; e^{-i\theta}) \; c_{-1}. \tag{177}$$

From the expression for c_{-1} given in (157), it turns out that:

$$\begin{aligned} \begin{pmatrix} f \\ g \end{pmatrix}^{\dagger} &= \frac{e^{i\theta}}{(\xi^2 + \tau^2 - \lambda^2)(\lambda w + i\xi + i\sqrt{\xi^2 - \lambda^2})} \\ &\times \begin{pmatrix} \lambda + w \; (-i\xi + \tau) & \quad e^{-i\theta}(i\xi + \tau + \lambda \; w) \end{pmatrix}. \end{aligned} \tag{178}$$

Replacing this expression into (165), we obtain:

$$\begin{aligned} d_{-1} = & \frac{e^{-t\sqrt{\xi^2-\lambda^2}}}{(\xi^2 + \tau^2 - \lambda^2) \; (w\lambda + i\xi + i\sqrt{\xi^2 - \lambda^2})} \\ & \times \begin{pmatrix} i(\xi+\sqrt{\xi^2-\lambda^2}) \; (\lambda-w(i\xi-\tau)) & ie^{-i\theta} \; (\xi+\sqrt{\xi^2-\lambda^2}) \; (w\lambda+i\xi+\tau) \\ \lambda \; e^{i\theta} \; (\lambda-w(i\xi-\tau)) & \lambda \; (w\lambda+i\xi+\tau) \end{pmatrix}. \end{aligned} \tag{179}$$

Now, taking into account (61), i.e.,

$$\tilde{d}_{-1} = -\oint_{\Gamma^-} d\tau \; e^{-i\tau u} d_{-1}(\theta, t, \xi, \tau, \lambda), \tag{180}$$

we finally get:

$$\begin{aligned} \tilde{d}_{-1} = & \pi i \frac{e^{-(u+t)\sqrt{\xi^2-\lambda^2}}}{\sqrt{\xi^2 - \lambda^2}(iw\lambda - \xi - \sqrt{\xi^2 - \lambda^2})} \\ \times & \begin{pmatrix} (\xi+\sqrt{\xi^2-\lambda^2}) \; (i\lambda+w(\xi+\sqrt{\xi^2-\lambda^2})) & e^{-i\theta}(\xi+\sqrt{\xi^2-\lambda^2}) \; (iw\lambda-\xi+\sqrt{\xi^2-\lambda^2}) \\ -i\lambda \; e^{i\theta}(i\lambda-w(\xi+\sqrt{\xi^2-\lambda^2})) & -i\lambda \; (iw\lambda-\xi+\sqrt{\xi^2-\lambda^2}) \end{pmatrix}. \end{aligned} \tag{181}$$

5.3 Calculation of the Green function

We are looking for the function $G_B(x,y)$ satisfying:

$$\begin{aligned} D_\alpha\ G_B(x,y) &= \delta(x,y), \\ B\ G_B(x,y)|_{x\in\partial\Omega} &= 0, \end{aligned} \tag{182}$$

where D_α and B, are given by equations (149) and (167) respectively. Now, some notation is in order:

$$\begin{gathered} x = (x_0, x_1) = (r\cos\theta, r\sin\theta), \\ X = x_0 + i\ x_1 = r\ e^{i\theta}, \\ \\ y = (y_0, y_1) = (\rho\cos\varphi, \rho\sin\varphi), \\ Y = y_0 + i\ y_1 = \rho\ e^{i\varphi}. \end{gathered} \tag{183}$$

It is easy to see that $G_B(x,y)$ can be written as

$$G_B(x,y) = e^{\alpha\gamma_5\phi(r)}\left[G_0(x,y) + h(x,y)\right]e^{\alpha\gamma_5\phi(\rho)}, \tag{184}$$

where

$$G_0(x,y) = \frac{1}{2\pi i}\frac{\not{x} - \not{y}}{(x-y)^2} \tag{185}$$

is the Green function of the operator $i\not{\partial}$ and $h(x,y)$ is a solution of the homogeneous equation

$$i\ \not{\partial}\ h(x,y) = 0, \tag{186}$$

to be determined through the boundary condition (166). Due to the geometry of the problem, one can make the ansatz

$$h(x,y) = \ G_0(x,\tilde{y})\ H(y), \tag{187}$$

where $\tilde{y}$ is related to y trough the inversion $\tilde{y} = y\frac{R^2}{\rho^2}$. Then, taking into account that

$$\frac{1}{r}\ \gamma_r\ G_0(\tilde{x},y) = -G_0(x,\tilde{y})\ \frac{1}{\rho}\ \gamma_\rho, \tag{188}$$

one finds:

$$H(y) = e^{2\alpha\gamma_5\phi(r)}\ \frac{R}{\rho}\ \frac{\left[(1+w^2)\ I + (1-w^2)\ \gamma_5\right]}{2w}\ \gamma_\rho. \tag{189}$$

Thus, the relevant Green function is given by

$$G_B(x,y) = \frac{1}{2\pi i} \begin{pmatrix} \frac{R\ w\ e^{\alpha(\phi(x)+\phi(y)-2\phi(R))}}{XY^*-R^2} & \frac{e^{\alpha(\phi(x)-\phi(y))}}{X-Y} \\ \frac{e^{-\alpha(\phi(x)-\phi(y))}}{(X-Y)^*} & \frac{R\ e^{-\alpha(\phi(x)+\phi(y)-2\phi(R))}}{w\ (XY^*-R^2)^*} \end{pmatrix}. \tag{190}$$

5.4 Evaluation of the determinant

With these elements at hand, we now go back to (153), and perform the calculation of the determinant .

From (190), one can see that

$$G_B(\theta, r, \varphi, r) \underset{\varphi\to\theta}{\sim} \text{diagonal matrix} + \frac{1}{2\pi \mathrm{i}\ \mathrm{r}\ (\theta-\varphi)}\ \gamma_\theta. \tag{191}$$

When replaced into (153), we get for the first term in the r.h.s.

$$tr\left\{A_\theta\ \gamma_\theta\ G_B(\theta,r,\varphi,r)\right\} \underset{\varphi\to\theta}{\sim} \frac{A_\theta}{\pi i\ r\ (\theta-\varphi)}. \tag{192}$$

For the second term in (153)

$$-\frac{1}{4\pi^2|x-y|}\int d\xi\ d\tau\ e^{i\xi\frac{(x-y)}{|x-y|}}\ c_{-1}(x,t;\frac{(\xi,\tau)}{|(\xi,\tau)|};\lambda=0) \underset{\varphi\to\theta}{\sim} \frac{-1}{2\pi i\ r\ (\theta-\varphi)}\ \gamma_\theta, \tag{193}$$

which exactly cancels the singularity of the Green function. (Notice that this Fourier transform must be understood in the sense of distributions). Therefore, the contribution of the first two terms in (153) vanishes.

As regards the third term,

$$\begin{aligned} &\frac{-1}{(2\pi)^2}\ tr\int \lim_{y\to x}\ \not{A}(t)\int_{|(\xi,\tau)|\geq 1} e^{i\xi(x-y)}\ c_{-2}(x,t;\xi,\tau;0)\ d\xi\ d\tau\ dx\ dt \\ &\quad = \frac{-\alpha}{2\pi^2}\ \lim_{y\to x}\int A_\theta^2\ d^2x\ \int_{|(\xi,\tau)|\geq 1} e^{i\xi(x-y)}\ \frac{(\tau^2-\xi^2)}{(\xi^2+\tau^2)^2}\ d\xi\ d\tau \qquad (194) \\ &\quad = \frac{-\alpha}{\pi}\int A_\theta^2\ d^2x\ \ \lim_{y\to x}\int_{|x-y|}^\infty J_2(u)\ \frac{du}{u}\ =\ \frac{-\alpha}{2\pi}\int A_\nu\ A_\nu\ d^2x\ . \end{aligned}$$

where $J_2(u)$ is the Bessel function of order two.

Now, the fourth term in (153) is:

$$\frac{-1}{(2\pi)^2}\ tr\int \not{A}(t)\int \frac{i}{2\pi}\int_\Gamma\ \ln\lambda\ \ c_{-2}(x,t;\frac{(\xi,\tau)}{|(\xi,\tau)|};\lambda)\ \ \frac{d\lambda}{\lambda}\ d\sigma_{\xi,\tau}\ dx\ dt$$

$$= \frac{-i\alpha}{4\pi^3}\int A_\theta^2\ d^2x\ \int_\Gamma\ \ \frac{\ln\lambda}{(\lambda^2-1)^2}\int\ (1-\lambda^2-2\xi^2)\ d\sigma_{\xi,\tau}\ \frac{d\lambda}{\lambda} \tag{195}$$

$$= \frac{i\alpha}{2\pi^2}\int A_\theta^2\ d^2x\ \ 2\pi i\ \int_0^\infty \frac{\mu\ d\mu}{(\mu^2+1)^2}\ \ =\ \frac{-\alpha}{2\pi}\int A_\nu\ A_\nu\ d^2x\ .$$

This term gives rise to a contribution identical to that of (194).

The last term in (153) is

$$\begin{aligned}&\frac{i}{(2\pi)^3}\ tr\int \not{A}(0)\ \sum_{\xi=\pm1}\int_\Gamma\ \ln\lambda\ \ \tilde{d}_{-1}(x,t;\frac{\xi}{|\xi|},t;\lambda)\ \frac{d\lambda}{\lambda}\ dx\ dt\\ &= \frac{i\ \Phi}{(2\pi)^2}\int_\Gamma\ \frac{u\ \ln\lambda}{(1+\ u^2\ \lambda^2)}[\lambda\ \sqrt{1+u^2}-i\sqrt{1-\lambda^2}]\ \frac{d\lambda}{\sqrt{1-\lambda^2}},\end{aligned} \tag{196}$$

where $u=(1-w^2)\ /\ 2w$. We choose the curve Γ as in Fig. 1.

Therefore, (196) reads

$$\begin{aligned}&-\frac{\Phi}{2\pi}\ u\ \int_0^\infty \frac{1}{(1-\ u^2\ \mu^2)}\left[\mu\ \ \frac{\sqrt{1\ +\ u^2}}{\sqrt{1\ +\ \mu^2}}\ -\ 1\right]d\mu\\ &= -\frac{\Phi}{4\pi}\left[\ln\left(\frac{\sqrt{1\ +\ u^2}\ +\ u\ \sqrt{1\ +\ \mu^2}}{\sqrt{1\ +\ u^2}\ -\ u\ \sqrt{1\ +\ \mu^2}}\ \ \frac{1\ -\ u\ \mu}{1\ +\ u\ \mu}\right)\right]_0^\infty\\ &= \frac{-\Phi}{4\pi}\ \ln\ w^2.\end{aligned} \tag{197}$$

Putting all pieces together ((194), (195) and (197), we finally find:

$$\begin{aligned}\ln Det(D)_B-\ln Det(i\ \not{\partial})_B &= -\frac{1}{2\pi}\int_\Omega A_\nu\ A_\nu\ d^2x\ \ -\frac{\Phi}{4\pi}\ln\ w^2\\ &= -\frac{1}{2\pi}\int_\Omega A_\nu\ A_\nu\ d^2x\ \ -\ \frac{1}{4\pi}\ \ln\ w^2\int_{\partial\Omega} A_\nu\ dx_\nu.\end{aligned} \tag{198}$$

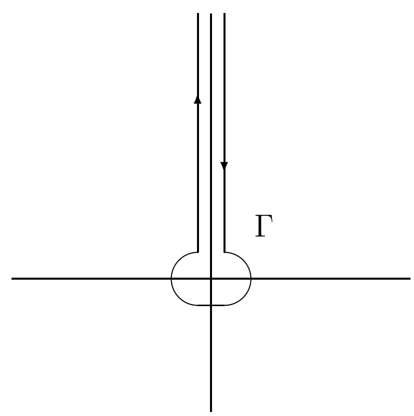

Figure 1: The contour Γ

The first term is the integral, restricted to the region Ω, of the same density appearing in the well known case of the whole plane [16]. The second term is well-defined for every $w \neq 0$, and vanishes for a null total flux, $\Phi = 0$. For $w = 0$, b_0 in (166) does not define an elliptic boundary problem, as discussed in Section 2. It is also interesting to notice that this term vanishes in the case of MIT bag boundary conditions, i.e., $w = \pm 1$.

This calculation is to be compared with the case of the compactified plane [8], where the determinant can be expressed in terms of just the kernel of the z-power of the operator analytically extended to $z = 0$, which is a local quantity. The presence of boundaries makes the evaluation more involved, since even in simple cases as the present (or the half plane treated in [6]), the knowledge of the Green function of the problem is needed.

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References

[1] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Math. Proc. Camb. Phil. Soc. **77**, 43 (1975); **78**, 405 (1975); **79**, 71 (1976).

[2] B. Booss, B., Bleecker, D.D.: Topology and Analysis. The Atiyah-Singer Index Formula and Gauge-Theoretic Physics. Springer-Verlag, New York, 1985.

[3] Booss, D., Wojciechowski, K.P. : Elliptic Boundary Problems for Dirac Operators. Birkhäuser, Boston, 1993.

[4] Calderón, A.P.: Lectures notes on pseudodifferential operators and elliptic boundary value problems, I. Publicaciones del I.A.M., Buenos Aires, 1976.

[5] Chodos, A., Jaffe, R.L., Johnson, K., Thorn, C.B., Weisskoft, V.F.: Phys. Rev. **D9**, 3471 (1974); A. Chodos, R.L. Jaffe, K. Johnson and C.B. Thorn: Phys. Rev. **D10**, 2599 (1974).

[6] Falomir, H., Muschietti, M.A., Santangelo, E.M.: J. Math. Phys. **31**, 989 (1990).

[7] Forman, R.: Functional Determinants and geometry. Invent. math. **88**, 447 (1987).

[8] Gamboa Saraví, R.E., Muschietti, M.A., Schaposnik F.A., Solomin, J.E.: Ann. Phys. **157**, 360 (1984).

[9] Gamboa Saraví, R.E., Muschietti, M.A., Schaposnik, F.A., Solomin, J.E.: J. Math. Phys. **26**, 2045 (1985).

[10] Gilkey, P.B., Smith,L.: Communications on Pure and Applied Mathematics, Vol. XXXVI, 85-131 (1983).

[11] Grubb, G., Seeley, R.T.: Developpements Asymptotiques pour l'operateur d'Atiyah-Patodi-Singer. C.R. Acad. Sci. Paris, **317**, serie I, p. 1123-1126 (1993);
Zeta and eta fuctions for Atiyah-Patodi-Singer operators, Copenh. Univ., Math. Dept, Prepint Ser., (1993), nro. 11;
Weakly parametric pseudodifferentials operators and Atiyah-Patodi-Singer boundary ploblems, Copenh. Univ., Math. Dept, Prepint Ser., (1993), nro 15.

[12] Hawking, S.W.: Comm. Math. Phys. **55**, 133 (1977).

[13] Hörmander, L.: The Analysis of Linear Partial Differential Operators III, Pseudo-Differential Operators. Springer-Verlag, Berlin Heidelberg, 1985.

[14] Moreno, E.F.: Phys. Rev. **D48**, 921 (1993).

[15] Ninomiya, M., Tan, C.I.: Nucl. Phys. **B257**, 199 (1985).

[16] Schwinger,J.: Phys. Rev. **82**, 664 (1951); **128**, 2425 (1962).

[17] Seeley, R.T.: The resolvent of an elliptic boundary problem. Am. J. Math. 91, 889-920 (1969).

[18] Seeley, R.T.: Analytic extension of trace associated with elliptic boundary problems. Am. J. Math. 91, 963-983 (1969).

[19] Stein, E.M.: Singular Integrals and Differentiability Properties of functions. Princeton University Press, Princeton, New Jersey, 1970.